FAST COLOR WAVELET-HAAR-HARTLEY-PROMETHEUS TRANSFORMS FOR IMAGE PROCESSING

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Abstract This paper present a new approach to the Color Fourier Transformation. Color image processing is investigated in this paper using an algebraic approach based on triplet (color) numbers. In the algebraic approach, each image color pixel is considered not as a 3D vector, but as a triplet (color) number. The so-called orthounitary transforms are introduced and used for color image processing. These transforms are similar to a fast orthogonal and unitary transforms. Simulations using the color Wavelet-Haar-Prometheus transforms on color image compression have also been performed.

Keywords: Clifford algebra, color images, color wavelet, edge detection, orthounitary transforms.

1. Introduction

Fourier analysis based on orthogonal and unitary transforms plays an important role in digital image processing. Transforms, notably the classical discrete Fourier transform, are extensively used in digital image filtering and in power spectrum estimation. Other Fourier transforms—e.g., the discrete cosine/sine transforms, wavelet transforms—are often employed in digital image compression. All the above-mentioned transforms are used in digital grey-level image processing. However, in recent years an increasing interest in color processing has been observed. Our approach to color image processing is in using so-called color triplet numbers [1]–[7] for color images and to operate directly on three-channel (RGB-valued) images as on single-channel triplet-valued images. In the classical approach every color is associated to a point of the 3D color

RGB vector space. In our approach, each image color pixel is considered not as a 3D RGB vector, but as a triplet (color) number.

A natural question that arises in our approach is the definition of color (RGB-channel) transforms that can be used efficiently in edge detection and digital image compression. The so-called orthounitary (triplet-valued or color-valued) Fourier transforms are introduced and are used for color image processing. These transforms are similar to fast orthogonal and unitary transforms. Therefore, fast algorithms for their computation can be easily constructed. Simulations of application of color transforms to color image compression have also been performed. The main contributions of this paper are: a) the definition and analysis of properties of the orthounitary (color) Fourier transforms (in particular, color Wavelet-Haar-Prometheus transforms); b) showing that the triplet (color) algebra can be used to solve color image processing problems in a natural and effective manner.

2. Color images

The aim of this section is to present algebraic models of the subjective perceptual color space. The color representation we are using is based on Young's theory (1802), asserting that any color can be visually reproduced by a proper combination of three colors, referred to as primary colors. The color image appears on the retina as a 3D vector-valued ((R, G, B)-valued) function

$$\mathbf{f}_{col}(\mathbf{x}) = \Big(f_R(\mathbf{x}), f_G(\mathbf{x}), f_B(\mathbf{x})\Big) = f_R(\mathbf{x})\mathbf{i}_R + f_G(\mathbf{x})\mathbf{i}_G + f_B(\mathbf{x})\mathbf{i}_B,$$

where $f_R(\mathbf{x}) = \int_\lambda s^{obj}(\mathbf{x}, \lambda)H_R(\lambda)d\lambda$, $f_G(\mathbf{x}) = \int_\lambda s^{obj}(\mathbf{x}, \lambda)H_G(\lambda)d\lambda$, and $f_B(\mathbf{x}) = \int_\lambda s^{obj}(\mathbf{x}, \lambda)H_B(\lambda)d\lambda$, $s^{obj}(\mathbf{x}, \lambda)$ is the color spectrum received from the object, $H_R(\lambda)$, $H_B(\lambda)$, $H_R(\lambda)$ are three photoreceptor (cone or sensor) sensitivity functions, λ is the wavelength and $\mathbf{i}_R := (1, 0, 0)$, $\mathbf{i}_G := (0, 1, 0)$, $\mathbf{i}_B := (0, 0, 1)$.

In our approach, each image color pixel is considered not as a 3D RGB vector, but as a triplet number in the following two forms (see [1] in this book): $\mathbf{f}_{col}(\mathbf{x}) = f_R(\mathbf{x})1_{col} + f_G(\mathbf{x})\varepsilon_{col} + f_B(\mathbf{x})\varepsilon_{col}^2$, $\mathbf{f}_{col}(\mathbf{x}) = f_{lu}(\mathbf{x})\mathbf{e}_{lu} + \mathbf{f}_{Ch}(\mathbf{x})\mathbf{E}_{Ch}$, where $\varepsilon^3 = 1$. The first and the second expressions are called the $\mathcal{A}_3(\mathbf{R}|1, \varepsilon_{col}, \varepsilon_{col}^2)$- and $\mathcal{A}_3(\mathbf{R}, \mathbf{C})$- *representations of color image*, respectively. Numbers of the form $\mathcal{C} = x + y\varepsilon_{col} + z\varepsilon_{col}^2$ are called the *triplet, 3-cycle,* or *color numbers.* Every color number $\mathcal{C} = x + y\varepsilon^1 + z\varepsilon^2$ is a linear combination $\mathcal{C} = x + y\varepsilon^1 + z\varepsilon^2 = a_{lu}\mathbf{e}_{lu} + \mathbf{z}_{Ch}\mathbf{E}_{Ch} = (a_{lu}, \mathbf{z}_{Ch})$ of the "scalar" $a_{lu}\mathbf{e}_{lu}$ and "complex" parts $\mathbf{z}_{Ch}\mathbf{E}_{Ch}$ in the idempotent basis $\{\mathbf{e}_{lu}, \mathbf{E}_{Ch}\}$. Real numbers $a_{lu} \in \mathbf{R}$ we will call *intensity (luminance) numbers*, and complex numbers $\mathbf{z}_{Ch} =$

$b + jc \in \mathbf{C}$ are called the *chromaticity numbers.* Thus $f_{lu}(\mathbf{x})$ is a real-valued (grey-level) image and $\mathbf{f}_{Ch}$ is a complex-valued (chromatic-valued) image.

A 2D discrete color image can be defined as a 2D array $\mathbf{f}_{col} := [\mathbf{f}_{col}(i,j)]_{i,j=1}^{N}$ i.e., as a 2D discrete $\mathcal{A}_3^{col}$-valued function in one of the following forms $\mathbf{f}_{col}(i,j) : \mathbf{Z}_N^2 \longrightarrow \mathcal{A}_3^{col}(1,\varepsilon^1,\varepsilon^2)$, $\mathbf{f}_{col}(i,j) : \mathbf{Z}_N^2 \longrightarrow \mathcal{A}_3^{col}(\mathbf{R},\mathbf{C})$, $\quad (i,j) \in \mathbf{Z}_N^2$. Here, every color pixel $\mathbf{f}_{col}(i,j)$ at position (i,j) is a color number of the type

$$\mathbf{f}_{col}(i,j) := f_r(i,j) + f_g(i,j)\varepsilon^1 + f_b(i,j)\varepsilon^2$$

or of the type

$$\mathbf{f}_{col}(i,j) := f_{lu}(i,j)\mathbf{e}_{lu} + f_{Ch}(i,j)\mathbf{E}_{Ch}.$$

The set of all such images forms N^2D Greaves-Hilbert space $\mathbb{L}(\mathbf{Z}_N^2, \mathcal{A}_3^{col}) = (\mathcal{A}_3^{col})^{N^2} = \mathbf{R}^{N^2}1 + \mathbb{R}^{N^2}\varepsilon^1 + \mathbf{R}^{N^2}\varepsilon^2 = \mathbf{R}^{N^2}\mathbf{e}_{lu} + \mathbf{C}^{N^2}\mathbf{E}_{Ch} = \mathbf{R}^{N^2} \oplus \mathbf{C}^{N^2}$, where $\mathbf{R}^{N^2}1$, $\mathbf{R}^{N^2}\varepsilon^1$, $\mathbb{R}^{N^2}\varepsilon^2$ are real N^2D Hilbert spaces of red, green, and blue images, respectively, $\mathbf{R}^{N^2}$ is the N^2D real space of gray-level images, and $\mathbf{C}^{N^2}$ is the N^2D complex space of chromaticity images.

A color linear operator $\mathfrak{L}_{2D} : \left(\mathcal{A}_3^{col}\right)^{N^2} \longrightarrow \left(\mathcal{A}_3^{col}\right)^{N^2}$, $\mathfrak{L}_{2D}[\mathbf{f}_{col}] = \mathbf{F}_{col}$ is said to be *orthounitary* if $\mathfrak{L}_{2D}^{-1} = \mathfrak{L}_{2D}^{*}$. Orthounitary operators preserve scalar product $\langle .|.\rangle$ and form orthounitary group transforms $\mathbb{OU}(\mathcal{A}_3^{col})$. This group is isomorphic to the direct sum of orthogonal and unitary groups $\mathbb{O}(\mathbb{R})\mathbf{e}_{lu} + \mathbb{U}(\mathbb{C})\mathbf{E}_{Ch}$ and every element has the representation $\mathfrak{L}_{2D} = \mathbf{O}_{2D}\mathbf{e}_{lu} + \mathcal{U}_{2D}\mathbf{E}_{ch}$, where $\mathbf{O}_{2D} \in \mathbb{O}(\mathbf{R})$ and $\mathcal{U}_{2D} \in \mathbb{U}(\mathbf{C})$ are orthogonal and unitary transforms, respectively. For color image processing we shall use separable $2D$ transforms. The orthounitary transform $\mathfrak{L}_{2D}[\mathbf{f}_{col}] = \mathbf{F}_{col}$ is called separable if it can be represented as $\mathbf{F}_{col} = \mathfrak{L}_{2D}[\mathbf{f}_{col}] = \mathfrak{L}_{1D}[\mathbf{f}_{col}]\mathfrak{M}_{2D}$, i.e. $\mathfrak{L}_{2D} = \mathfrak{L}_{1D} \otimes \mathfrak{M}_{1D}$ is the tensor product of two $1D$ orthounitary transforms of the form $\mathfrak{L}_{2D} = \mathfrak{L}_{1D} \otimes \mathfrak{L}_{1D} = (\mathbf{O}_1 \otimes \mathbf{O}_2)\mathbf{e}_{lu} + (\mathcal{U}_1 \otimes \mathcal{U}_2)\mathbf{E}_{ch}$, where $\mathbf{O}_1, \mathbf{O}_2$ and $\mathcal{U}_1, \mathcal{U}_2$ are 1D orthogonal and unitary transforms, respectively. Thus, we can obtain any orthounitary transform, using any two pairs of orthogonal $\mathbf{O}_1, \mathbf{O}_2$ and unitary transforms $\mathcal{U}_1, \mathcal{U}_2$. In this work we shall use one pair of orthogonal and unitary transforms where $\mathbf{O}_1 = \mathbf{O}_2 = \mathbf{O}$ and $\mathcal{U}_1 = \mathcal{U}_2 = \mathcal{U}$. In this case we obtain a wide family of orthounitary transforms of the form $\mathfrak{L}_{2D} = (\mathbf{O} \otimes \mathbf{O})\mathbf{e}_{lu} + (\mathcal{U} \otimes \mathcal{U})\mathbf{E}_{ch}$ using different 1D orthogonal transforms. In this work we shall use the more simple orthounitary transforms $\mathfrak{L} = \mathbf{O} \cdot \mathbf{e}_{lu} + \mathcal{U} \cdot \mathbf{E}_{Ch}$ with $\mathcal{U} = \mathbf{O} \cdot \mathbf{diag}(z_0, z_1, \ldots, z_{N-1})$, where $\mathbf{diag}(z_0, z_1, \ldots, z_{N-1})$ is a diagonal matrix of chromatic (complex) numbers. In this case we have $\mathfrak{L} = \mathbf{O} \cdot \mathbf{e}_{lu} + \mathbf{O} \cdot \mathbf{diag}(z_0, z_1, \ldots, z_{N-1}) \cdot \mathbf{E}_{Ch} =$

$\mathbf{O}\cdot\mathbf{diag}(\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{N-1})$, where $\mathcal{C}_k := 1\mathbf{e}_{lu} + z_k\mathbf{E}_{Ch}$, $k = 0, 1, \ldots, N-1$, are color (triplet) numbers. Hence, our orthounitary transforms are represented as the product of an orthogonal transform $\mathbf{O}$ and a triplet-valued (color) diagonal matrix $\mathbf{diag}(\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{N-1})$. A large number of orthounitary transforms can be devised by appropriate choice of the orthogonal transform $\mathbf{O}$ and the diagonal transform parameters $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{N-1}$.

3. Color Wavelet-Haar-Prometheus transforms

One of the aims of this paper is to define three-channel transforms (the so-called color Fourier transforms) that could eventually be used in digital color image compression. We have experimented with the so-called *color Wavelet-Haar* $\mathbf{WH}_{2^n}$, *color Wavelet-Haar-Prometheus* $\mathbf{WHP}_{2^n}$, *Wavelet-Haar-Hartley* $\mathbf{WHH}_{3^n}$, and *Wavelet-Haar-Hartley-Prometheus* $\mathbf{WHHP}_{3^n}$, *transforms* [8]–[9].

Orthogonal and unitary Wavelet-Haar and Wavelet-Haar-Prometheus transforms have the factorizations: $\mathbf{WHP}_{2^n} = \mathbf{WH}_{2^n}\Delta_{2^n}$, $\mathcal{WHP}_{2^n} = \mathcal{WH}_{2^n}\Delta_{2^n}$, and

$$\mathbf{WH}_{2^n} = \prod_{i=1}^{n}\left[(\mathbf{I}_{2^{n-i}} \overset{\circ}{\otimes} \mathbf{F}_2) \oplus \mathbf{I}_{2^n - 2^{n-i+1}}\right], \qquad (1)$$

$$\mathcal{WH}_{2^n} = \prod_{i=1}^{n}\left[(\mathbf{I}_{2^{n-i}} \overset{\circ}{\otimes} \mathcal{F}_2) \oplus \mathbf{I}_{2^n - 2^{n-i+1}}\right], \qquad (2)$$

respectively, where

$$\mathbf{I}_{2^{n-i}} \overset{\circ}{\otimes} \mathbf{F}_2 = C\left[\frac{\mathbf{I}_{2^{n-i}} \otimes [1 \quad 1]}{\mathbf{I}_{2^{n-i}} \otimes [1 - 1]}\right], \; \mathbf{I}_{2^{n-i}} \overset{\circ}{\otimes} \mathcal{F}_2 = C\left[\frac{\mathbf{I}_{2^{n-i}} \otimes [1 \quad \omega_3]}{\mathbf{I}_{2^{n-i}} \otimes [1 - \omega_3]}\right]$$

are the generalized tensor products of the identity matrix $\mathbf{I}_{2^{n-i}}$ with $\mathbf{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\mathcal{F}_2 = \begin{bmatrix} 1 & \omega_3 \\ 1 & -\omega_3 \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & \\ & \omega_3 \end{bmatrix}$, respectively. Here, $c = \frac{\sqrt{2}}{2}$, $\mathbf{F}_2$ is the classical Walsh transform and $\mathcal{F}_2$ is the complex ω_3-deformed Walsh transform, where $\omega_3 = \sqrt[3]{1} = e^{2\pi i/3}$ and Δ_{2^n} is a diagonal matrix, whose diagonal elements form the Shapiro (± 1)-sequence. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the binary representation of the number of the αth row of Δ_{2^n}, where $\alpha_i \in \mathbf{Z}_2$, then for diagonal elements $\Delta_{\alpha,\alpha}$ we have the expression $\Delta_{\alpha,\alpha} = (-1)^{\sum_{i=1}^{n-1}\alpha_i\alpha_{i+1}}$. The quantity $b(\alpha) = \sum_{i=1}^{n-1}\alpha_i\alpha_{i+1}$ is the number of occurrences of the digital block $B = (11)$ in the binary representation $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of the number α.

Using two pairs $(\mathbf{WH}_{2^n}, \mathcal{WH}_{2^n})$ and $(\mathbf{WHP}_{2^n}, \mathcal{WHP}_{2^n})$ of fast orthogonal and unitary Haar-Wavelet transforms (1–2), we construct 1D color Wavelet-Haar and Wavelet-Haar-Prometheus 2^n-point fast transforms as

$$\mathfrak{WH}_{2^n} = \mathbf{WH}_{2^n} \cdot \mathbf{e}_{lu} + \mathcal{WH}_{2^n} \cdot \mathbf{E}_{Ch} = \left(\prod_{i=1}^{n} \left[(\mathbf{I}_{2^{n-i}} \overset{\circ}{\otimes} \mathfrak{F}_2) \oplus \mathbf{I}_{2^n - 2^{n-i+1}} \right] \right), \tag{3}$$

$$\mathfrak{WHP}_{2^n} = \mathfrak{WH}_{2^n} \Delta_{2^n} = \Big(\mathbf{WHP}_{2^n} \cdot \mathbf{e}_{lu} + \mathcal{WHP}_{2^n} \cdot \mathbf{E}_{Ch} \Big) \Delta_{2^n} =$$

$$= \left(\prod_{i=1}^{n} \left[(\mathbf{I}_{2^{n-i}} \overset{\circ}{\otimes} \mathfrak{F}_2) \oplus \mathbf{I}_{2^n - 2^{n-i+1}} \right] \right) \Delta_{2^n}, \tag{4}$$

where $\mathfrak{F}_2 = \mathbf{F}_2 \cdot \mathbf{e}_{lu} + \mathcal{F}_2 \cdot \mathbf{E}_{Ch} =$

$$\tfrac{\sqrt{2}}{2} \left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \mathbf{e}_{lu} + \begin{bmatrix} 1 & \omega_3 \\ i & -\omega_3 \end{bmatrix} \cdot \mathbf{E}_{Ch} \right) = \tfrac{\sqrt{2}}{2} \left[\begin{array}{c|c} 1 & \varepsilon \\ \hline 1 & -\varepsilon \end{array} \right].$$

We see that the color Haar-Wavelet transform has the same fast algorithm as the orthogonal and unitary transforms in Eqs. (1). Note that the product of ε with a color pixel $\mathbf{f}_{col} = (f_R, f_G, f_B) = f_R 1 + f_G \varepsilon^1 + f_B \varepsilon^2$ is realized without multiplications as the right shift of color components $\varepsilon \mathbf{f}_{col} = \varepsilon(f_R, f_G, f_B) = \varepsilon(f_R 1 + f_G \varepsilon^1 + f_B \varepsilon^2) = (f_B 1 + f_R \varepsilon^1 + f_B \varepsilon^2) = (f_B, f_R, f_G)$. In this case the situation is the same as for Number Theoretical Transforms.

The next example of color (orthounitary) Haar-like wavelet transforms is based on Haar-Hartley $\mathbf{H}_3$ (3×3)-transforms. Using these transforms we construct an "elementary" three-point color transform of the following form in the $\mathcal{A}_3(\mathbf{R}, \mathbf{C})$-algebra: $\mathfrak{HF}_3 = \mathbf{H}_3 \cdot \mathbf{e}_{lu} + \mathcal{F}_3 \cdot \mathbf{E}_{Ch} =$

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & h_1 & h_2 \\ 1 & h_2 & h_1 \end{bmatrix} \mathbf{e}_{lu} + \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & h_1 \omega_3^1 & h_2 \omega_3^2 \\ 1 & h_2 \omega_3^1 & h_1 \omega_3^2 \end{bmatrix} \mathbf{E}_{Ch} =$$

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & h_1 \varepsilon^1 & h_2 \varepsilon^2 \\ 1 & h_2 \varepsilon^1 & h_1 \varepsilon^2 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & h_1 & h_2 \\ 1 & h_2 & h_1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & \varepsilon^1 & \\ & & \varepsilon^2 \end{bmatrix}, \tag{5}$$

where $h_1 := \operatorname{cas}\left(\frac{2\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right)$, $h_2 := \operatorname{cas}\left(\frac{2\pi 2}{3}\right) = \cos\left(\frac{2\pi 2}{3}\right) + \sin\left(\frac{2\pi 2}{3}\right)$ and $\omega_3^1 := \operatorname{cis}\left(\frac{2\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)$, $\omega_3^2 := \operatorname{cis}\left(\frac{2\pi 2}{3}\right) = \cos\left(\frac{2\pi 2}{3}\right) + i \sin\left(\frac{2\pi 2}{3}\right)$. We use the orthogonal and unitary Wavelet-Haar-Hartley and Wavelet-Haar-Hartley-Prometheus 3^n-point transforms that follow: $\mathbf{WHP}_{3^n} = \mathbf{WH}_{3^n}\Delta_{3^n}$, $\mathcal{WHP}_{3^n} = \mathcal{WH}_{3^n}\Delta_{3^n}$, and

$$\mathbf{WH}_{3^n} = \prod_{i=1}^{n}\left[(\mathbf{I}_{3^{n-i}} \overset{\circ}{\otimes} \mathbf{F}_3) \oplus \mathbf{I}_{3^n - 3^{n-i+1}}\right], \tag{6}$$

$$\mathcal{WH}_{3^n} = \prod_{i=1}^{n}\left[(\mathbf{I}_{3^{n-i}} \overset{\circ}{\otimes} \mathcal{F}_3) \oplus \mathbf{I}_{3^n - 2^{n-i+1}}\right], \tag{7}$$

respectively, where

$$\mathbf{I}_{3^{n-i}} \overset{\circ}{\otimes} \mathbf{F}_3 = \begin{bmatrix} \mathbf{I}_{3^{n-i}}\otimes[1 \;\; 1 \;\; 1 \;] \\ \hline \mathbf{I}_{3^{n-i}}\otimes[1 \;\; h_1 \;\; h_2 \;] \\ \hline \mathbf{I}_{3^{n-i}}\otimes[1 \;\; h_2 \;\; h_1 \;] \end{bmatrix}, \; \mathbf{I}_{3^{n-i}} \overset{\circ}{\otimes} \mathcal{F}_3 = \begin{bmatrix} \mathbf{I}_{3^{n-i}}\otimes[1 \;\; \omega_3^1 \;\; \omega_3^2 \;] \\ \hline \mathbf{I}_{3^{n-i}}\otimes[1 \;\; h_1\omega_3^1 \;\; h_2\omega_3^2 \;] \\ \hline \mathbf{I}_{3^{n-i}}\otimes[1 \;\; h_2\omega_3^1 \;\; h_1\omega_3^2 \;] \end{bmatrix}$$

are the generalized tensor products of the identity matrix $\mathbf{I}_{3^{n-i}}$ with

$$\mathbf{H}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & h_1 & h_2 \\ 1 & h_2 & h_1 \end{bmatrix} \quad \text{and} \quad \mathcal{H}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & h_1\omega_3^1 & h_2\omega_3^2 \\ 1 & h_2\omega_3^1 & h_1\omega_3^2 \end{bmatrix},$$

respectively, and Δ_{3^n} is a diagonal matrix whose diagonal elements form the 3-point Shapiro ω_3-sequence. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the 3-ary representation of the number in the αth row of Δ_{3^n}, where $\alpha_i \in \mathbf{Z}_2$, then for diagonal elements $\Delta_{\alpha,\alpha}$ we have the expression $\Delta_{\alpha,\alpha} = \omega_3^{\sum_{i=1}^{n-1}\alpha_i\alpha_{i+1}}$.

Using these fast orthogonal and unitary Wavelet-Haar-Hartley-Prometheus transforms (6–7) we construct 1D fast color Wavelet-Haar-Hartley-Prometheus 3^n-point transforms by $\mathfrak{WHP}_{3^n} = \mathfrak{WH}_{3^n}\Delta_{3^n}$ and

$$\mathfrak{WH}_{3^n} = \left(\prod_{i=1}^{n}\left[(\mathbf{I}_{3^{n-i}} \overset{\circ}{\otimes} \mathfrak{HF}_3) \oplus \mathbf{I}_{3^n - 3^{n-i+1}}\right]\right), \tag{8}$$

where $\mathfrak{HF}_3$ is the color 3-point Hartley transform (5).

4. Edge detection and compression of color images

One of the primary applications of this work could be in edge detection and color image compression. For edge detection, we convolve the color (3×3)-masks $\mathbf{m}_{col}(i,j)$ with a color image $\mathbf{f}_{col}(i,j)$ of size $N \times N$:

$$\widehat{f}(i,j) = \sum_{(i,j)\in\mathbf{Z}_N^2} m_{col}(k,l)\mathbf{f}_{col}(i-k, j-l).$$

We use color Prewitt's-like masks for detection of horizontal, vertical, and diagonal edges. As entries instead of real numbers these masks have

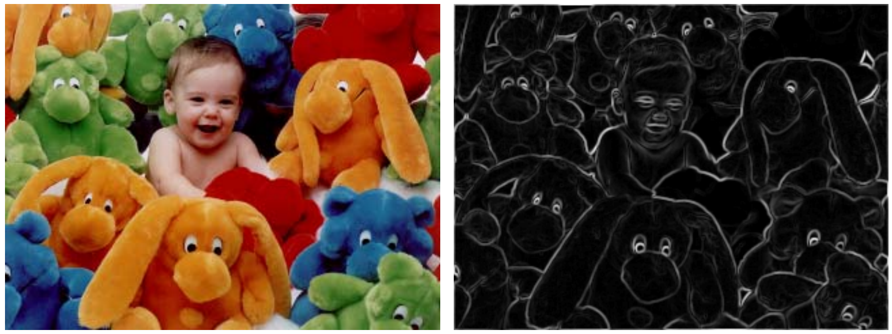

Figure 1. Color edge detector. Left: original image, right: detected edges.

triplet numbers:

$$\mathbf{m}_{col}^{H} = \begin{bmatrix} 1 & \varepsilon & \varepsilon^2 \\ 0 & 0 & 0 \\ -1 & -\varepsilon & -\varepsilon^2 \end{bmatrix}, \quad \mathbf{m}_{col}^{V} = \begin{bmatrix} 1 & 0 & -1 \\ \varepsilon & 0 & -\varepsilon \\ \varepsilon^2 & 0 & -\varepsilon^2 \end{bmatrix},$$

$$\mathbf{m}_{col}^{LD} = \begin{bmatrix} \varepsilon & \varepsilon^2 & 0 \\ 1 & 0 & -\varepsilon^2 \\ 0 & -1 & -\varepsilon \end{bmatrix}, \quad \mathbf{m}_{col}^{RD} = \begin{bmatrix} 0 & 1 & \varepsilon \\ -1 & 0 & \varepsilon^2 \\ -\varepsilon & -\varepsilon^2 & 0 \end{bmatrix}.$$

The effect of the masks in homogenous and nonhomogenous color regions differs substantially. Let us analyze both cases in detail. At any position (i, j) of a homogenous color region after convolution we get

$$\widehat{\mathbf{f}}_{col} = \begin{bmatrix} f_R^{11} + f_B^{12} + f_G^{13} \\ f_G^{11} + f_R^{12} + f_B^{13} \\ f_B^{11} + f_G^{12} + f_R^{13} \end{bmatrix} - \begin{bmatrix} f_R^{31} + f_B^{32} + f_G^{33} \\ f_G^{31} + f_R^{32} + f_B^{33} \\ f_B^{31} + f_G^{32} + f_R^{33} \end{bmatrix} = 0,$$

since, for all 9 pixels we have $\mathbf{f}_{col}^{11} = \mathbf{f}_{col}^{12} = \mathbf{f}_{col}^{13} = \mathbf{f}_{col}^{21} = \mathbf{f}_{col}^{22} = \mathbf{f}_{col}^{23} = \mathbf{f}_{col}^{31} = \mathbf{f}_{col}^{32} = \mathbf{f}_{col}^{33} =$ color const $= \mathcal{C}$. For horizontal nonhomogenous color regions we have $\mathbf{f}_{col}^{11} = \mathbf{f}_{col}^{12} = \mathbf{f}_{col}^{13} = \mathcal{C}_1 = a_{lu}^1 \mathbf{e}_{lu} + bfz_{ch}^1 \mathbf{E}_{ch}$, and $\mathbf{f}_{col}^{31} = \mathbf{f}_{col}^{32} = \mathbf{f}_{col}^{33} = \mathcal{C}_2 = a_{lu}^2 \mathbf{e}_{lu} + \mathbf{z}_{ch}^2 \mathbf{E}_{ch}$. Hence, $\widehat{\mathbf{f}}_{col} =$

$$= \begin{bmatrix} f_R^{11} + f_B^{12} + f_G^{13} \\ f_G^{11} + f_R^{12} + f_B^{13} \\ f_B^{11} + f_G^{12} + f_R^{13} \end{bmatrix} - \begin{bmatrix} f_R^{31} + f_B^{32} + f_G^{33} \\ f_G^{31} + f_R^{32} + f_B^{33} \\ f_B^{31} + f_G^{32} + f_R^{33} \end{bmatrix} = \begin{bmatrix} a_{lu}^1 \\ a_{lu}^1 \\ a_{lu}^1 \end{bmatrix} - \begin{bmatrix} a_{lu}^2 \\ a_{lu}^2 \\ a_{lu}^2 \end{bmatrix} = \Delta a_{lu} \mathbf{e}_{lu}.$$

Fig. 1 shows the result of color edge detection.

We have performed a number of simulations on the use of orthounitary transforms in the area of color image compression. We have experimented both with $2^n \times 2^n$ and $3^n \times 3^n$-pixel images by using $\mathfrak{WHP}_{2^n}$,

$\mathfrak{WHP}_{3^n}$, and $\mathbf{WHP}_{2^n}$, $\mathbf{WHP}_{3^n}$ transforms. Figures 2–3 illustrate the $\mathfrak{WHP}_{3^n}$ and $\mathbf{WHP}_{2^n}$, $\mathbf{WHP}_{3^n}$ transforms of the color image "BABOON" after the first iterations of the fast algorithms of these transformations. Examples of 2D color histrograms of chromaticity planes for different spectrums are shown in Fig. 4.

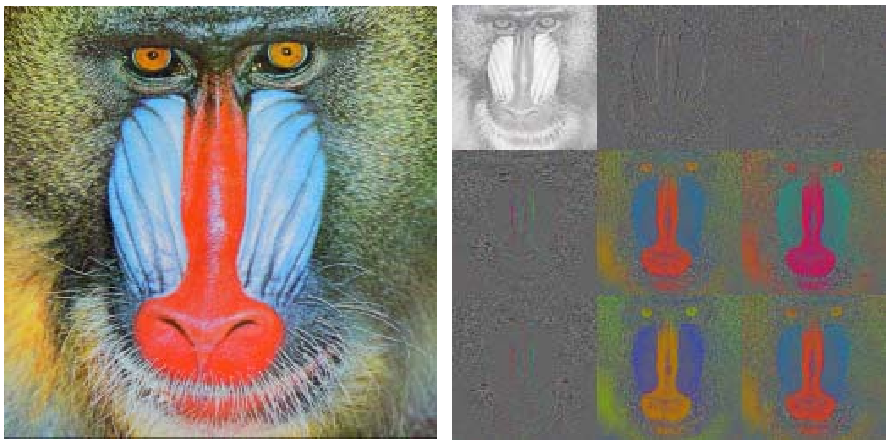

Figure 2. Left: original "BABOON" image, right: wavelet spectrum of "BABOON" after the first iterations of $\mathfrak{WHP}_{3^n}$.

5. Conclusion

A system of color-valued 2D basis functions has been defined in this paper. This system can be used to obtain color orthounitary Fourier transforms and series analyses of color images. Properties of the color Fourier transforms are presented. It is shown that such color series have properties similar to the classical orthogonal Fourier series. A family of discrete color orthounitary 2D Fourier transforms has also been presented that can be used in color image compression. In particular, the color Wavelet-Haar-Prometheus transforms are defined and used to obtain the color Wavelet-Haar-Prometheus series.

The analysis presented in this paper provides a very general framework for the definition of other multicolor transforms based on multiplet hypercomplex numbers. The derivation of such multicolor transforms is the subject of ongoing research. The motivation of this ongoing re-

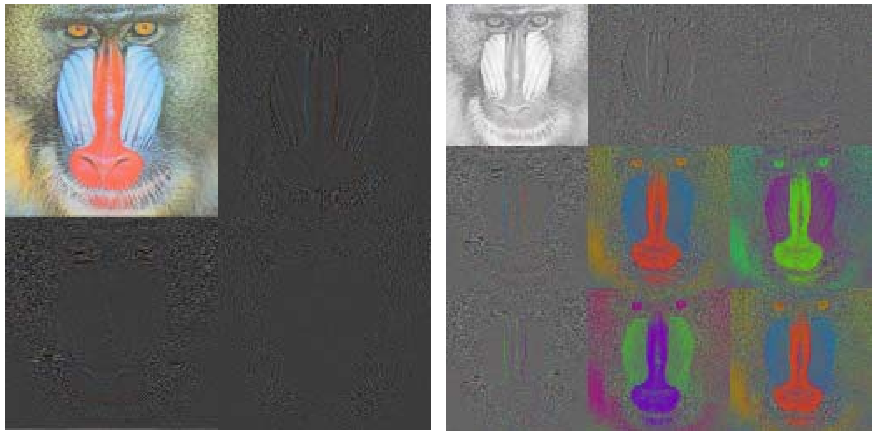

Figure 3. Wavelet spectrum of "BABOON" after the first iterations. Left: $\mathbf{WH}_{2^n}$, right: $\mathfrak{WH}_{3^n}$.

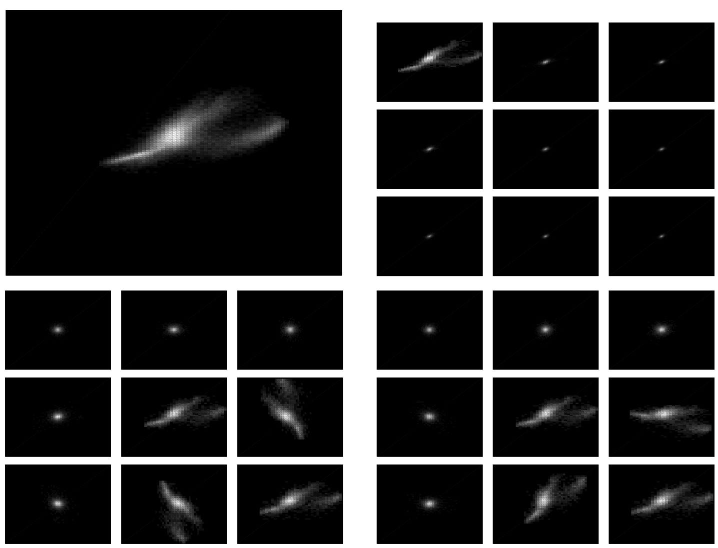

Figure 4. Examples of 2D histograms of the chromaticity plane. From top to bottom from left to right: original "BABOON" image, $\mathbf{WH}_3$ spectrum, $\mathfrak{WH}_3$ spectrum, $\mathfrak{WHP}_3$ spectrum.

search is to define multicolor transforms that can be used efficiently in multicolor satellite image compression.

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