GROUP THEORY IN RADAR AND SIGNAL PROCESSING

William Moran
Department of Electrical and Electronic Engineering
The University of Melbourne, Victoria 3010 Australia
b.moran@ee.mu.oz.au

Jonathan H. Manton
Department of Electrical and Electronic Engineering
The University of Melbourne, Victoria 3010 Australia
j.manton@ee.mu.oz.au

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1. Introduction

This paper describes some key mathematical ideas in the theory of radar from a group theoretic perspective. The intention is to elucidate how radar theory motivates interesting ideas in representation theory and, conversely, how representation theory affords a better understanding of the inherent limitations of radar. Although most of the results presented here can be found in (Wilcox, 1991) and (Miller, 1991), there are significant differences in the selection and presentation of material. Moreover, compared with (Wilcox, 1991), (Miller, 1991) and (Moran, 2001), greater emphasis is placed here on the group theoretic approach, and in particular, its ability to arrive quickly and succinctly at basic results about radar.

Central to radar theory is the ambiguity function. Specifically, corresponding to any waveform $\mathbf{w}(t)$ is a two dimensional function $A_{\mathbf{w}}(t, f)$, called the ambiguity function, which measures the ability of that particular waveform to allow the radar system to estimate accurately the location and velocity of the target. Some waveforms perform better than others, and it is the challenge of radar engineers to design waveforms with desirable ambiguity functions while simultaneously meeting the many other design criteria which impose constraints on the set of feasible waveforms.

A particularly elegant way of studying the ambiguity function is to write it in the form $A_{\mathbf{w}}(t,f) = \langle \mathbf{w}, \rho_{(t,f)}\mathbf{w}\rangle_{\mathsf{L}^2(\mathbf{R})}$ where $\rho_{(t,f)}$ is an operator acting on $\mathsf{L}^2(\mathbf{R})$. In fact, $\rho_{(t,f)}$ turns out to be a very special type of operator; it is an irreducible unitary multiplier representation of the additive group $\mathbf{R}^2$. It is here that group representation theory enters the picture. Functions $A : G \to \mathbf{C}$ of the form $A(g) = \langle \mathbf{w}, \rho_g \mathbf{v}\rangle$ over some group G, where ρ_g is a representation of G, are sometimes known as *special functions* in the literature. Importantly, most if not all interesting facts about special functions can be deduced from a study of the group representation ρ_g.

After defining the ambiguity function in Section 2, a brief introduction to representation theory is presented in Section 3. A feature of this presentation is that multiplier representations, along with their connections to ordinary representations and projective representations, are highlighted. Whereas it is customary to study the ambiguity function via the representation theory of the Heisenberg group, this paper studies instead the relevant multiplier representation of $\mathbf{R}^2$. Although both approaches are equivalent, the authors believe the multiplier representation approach is the more natural.

Section 4 derives fundamental facts about the multiplier representation theory of $\mathbf{R}^2$ and how it relates to the ambiguity function. A somewhat novel contribution is a generalised version of Moyal's identity (Theorem 2), whose proof has a more direct and intuitive flavour than that of previous proofs of Moyal's identity. Also covered in Section 4 are the various ways of realising the representation $\rho_{(t,f)}$ on different Hilbert spaces. These representations are equivalent to each other, but depending on the problem at hand, some spaces can be easier to work in than others.

The results in Section 4 are applied in Section 5 to answer several questions about ambiguity functions. A generating function approach is provided for finding explicit formulae for the ambiguity functions of Hermites. It is proved that the Hermite waveforms have the distinguishing property of having rotationally symmetric ambiguity functions. The potential benefits of hypothetically being able to use multiple waveforms are touched upon too.

Finally, Section 6 concludes by stating that the ambiguity function studied in this paper is the narrow band ambiguity function and is an approximation, albeit a good one in many situations, to what is known as the wide band ambiguity function. It is explained that the latter also can be studied from a group theoretic perspective, but that such a study is not undertaken here.

2. How a Radar Works

At the simplest level a radar transmits a waveform $\mathbf{w}(t)$ which is then reflected from the scene. In fact, the waveform is modulated onto (that is, multiplied by) a much higher (carrier) frequency sinusoid.

The reflection is picked up at the radar and processed to produce a picture of the scene. We assume for the purposes of this discussion that the radar is looking entirely in one direction. We can think of the scene for the moment as a collection of scatterers at various distances r_k and moving with velocities v_k relative to the radar. We measure distances in units of time — the time needed for light to travel that distance — and we measure the velocities in multiples of the speed of light. This prevents the proliferation of c's (the speed of light) in the formulae. Thus the scene can be regarded as a linear combination

$$\text{scene}(t) = \sum_k c_k \, \delta(t - r_k, v - v_k) \tag{1}$$

where $\delta(t, v)$ is the "delta" function — that is a point mass — at the origin of the "range-Doppler" (t, v) plane. The term c_k is the (complex) reflectivity of the scatterer.

Before proceeding further we remark that because radar (unlike conventional light based viewing systems) is able to keep track of the phase of the signal, or at least phase changes in it, processing in radar is done in the complex domain. A complex waveform can be modulated onto a carrier by using a phase shift to represent the argument of the waveform. The presence of the high frequency carrier and a stable oscillator producing the sinusoid means that the radar can detect phase shifts and is capable of producing a reasonably good approximation to the Hilbert transform of the return. As a result, radar engineers work in the complex domain and assume that their signals are complex, the argument corresponding to a phase shift in the carrier.

The signal returned to the receiver — and for simplicity we assume that the transmitter and receiver are collocated — is then the convolution of the transmit signal and the scene. That is,

$$\text{ret}(t) = \sum_k c_k \mathbf{w}(t - 2r_k) e^{4\pi i v_k f_c t}, \tag{2}$$

where f_c is the carrier frequency. The 2's have appeared because we are considering the round-trip of the signal rather than the one-way trip. The return is then (after stripping off the high frequency carrier) correlated with another (or the same) waveform $\mathbf{v}(t)$. The result is

$$\begin{aligned} \operatorname{proc}(t) &= \int \operatorname{ret}(\tau)\overline{\mathbf{v}(\tau - t)}\, d\tau \\ &= \sum_k c_k \int \mathbf{w}(\tau - 2r_k) e^{4\pi i v_k f_c \tau} \overline{\mathbf{v}(\tau - t)}\, d\tau, \end{aligned} \tag{3}$$

which is a linear combination over the individual scatterers, as one would expect. The expression

$$\int \mathbf{w}(\tau - 2r) e^{4\pi i v f_c \tau} \overline{\mathbf{v}(\tau - t)}\, d\tau \tag{4}$$

tells all about the way in which the radar behaves. Once we know this function and the scene, we can reconstruct the return. Of course this is a gross simplification but for the purposes of this paper it will be enough. It is important to mention that the key issue for the radar engineer is to do the inverse problem: given the return, how to extract the scene.

Briefly, we mention that there is scope for extraction of information from multiple waveforms rather than the single waveforms discussed here. Even this situation can be treated as that of one very long pulse and the expression (4) continues to be relevant.

By a change of variable and a renormalization of the units of measurement of the velocity, equation (4) becomes

$$A_{\mathbf{w},\mathbf{v}}(t, f) = \int \mathbf{w}(\tau) e^{2\pi i f \tau} \overline{\mathbf{v}(\tau - t)}\, d\tau. \tag{5}$$

This is the *radar ambiguity function*, or at least one form of it, and is the focus of the study of this paper.

At this point we impose some mathematical constraints on the situation. The functions $\mathbf{w}$ and $\mathbf{v}$ are always assumed to be “finite energy” signals; that is, they belong to $\mathsf{L}^2(\mathbf{R})$. It is clear then that the integrand is integrable and indeed, by the Cauchy-Schwarz Inequality, that

$$|A_{\mathbf{w},\mathbf{v}}(t, f)| \leq \|\mathbf{w}\|_{\mathsf{L}^2(\mathbf{R})} \|\mathbf{v}\|_{\mathsf{L}^2(\mathbf{R})}, \qquad (t, f) \in \mathbf{R}^2, \tag{6}$$

for all $\mathbf{v}, \mathbf{w} \in \mathsf{L}^2(\mathbf{R})$. The expression

$$\rho_{(t,f)}(\mathbf{v})(\tau) = e^{-2\pi i f \tau} \mathbf{v}(\tau - t) \tag{7}$$

is clearly significant in the theory of the ambiguity function. We note that $\rho_{(t,f)}$ is a unitary operator in $\mathsf{L}^2(\mathbf{R})$, and that

$$A_{\mathbf{w},\mathbf{v}}(t, f) = \left\langle \mathbf{w}, \rho_{(t,f)} \mathbf{v} \right\rangle_{\mathsf{L}^2(\mathbf{R})}. \tag{8}$$

Moreover

$$
\begin{aligned}
\rho_{(t_1,f_1)}\rho_{(t_2,f_2)}(\mathbf{v})(\tau) &= e^{-2\pi i f_1 \tau}\rho_{(t_2,f_2)}(\mathbf{v})(\tau - t_1) \\
&= e^{-2\pi i f_1 \tau} e^{-2\pi i f_2(\tau - t_1)}(\mathbf{v})(\tau - t_1 - t_2) \\
&= e^{2\pi i f_2 t_1}\rho_{(t_1+t_2,f_1+f_2)}(\mathbf{v})(\tau),
\end{aligned} \tag{9}
$$

that is,

$$
\rho_{(t_1,f_1)}\rho_{(t_2,f_2)} = e^{2\pi i f_2 t_1}\rho_{(t_1+t_2,f_1+f_2)}. \tag{10}
$$

This makes ρ a *multiplier representation* of $\mathbf{R}^2$ with *multiplier*

$$
\sigma((t_1, f_1), (t_2, f_2)) = e^{-2\pi i t_1 f_2} \tag{11}
$$

because

$$
\rho_{(t_1+t_2,f_1+f_2)} = \sigma((t_1, f_1), (t_2, f_2))\, \rho_{(t_1,f_1)}\rho_{(t_2,f_2)}. \tag{12}
$$

Our aim is to make sense of these ideas and examine their consequences.

3. Representations

Since representations, or at least multiplier representations, play a role in the radar ambiguity function, we spend some time discussing them; first ordinary representations and then multiplier representations. The abstract theory of representations of groups goes as follows. Let G be a locally compact group. Such a group has a Haar measure m_G, which is invariant under left translation:

$$
\forall h \in G, \quad \int_G F(hg)\, dm_G(g) = \int_G F(g)\, dm_G(g), \tag{13}
$$

for any integrable function F on G.

A (unitary) *representation* of G is a continuous homomorphism

$$
\pi : G \to \mathcal{U}(\mathfrak{H}) \tag{14}
$$

into the unitary group of a Hilbert space $\mathfrak{H}$. It will be convenient to write the image of $g \in G$ under this map as π_g instead of the more conventional $\pi(g)$. Continuity means, in this case, that the maps

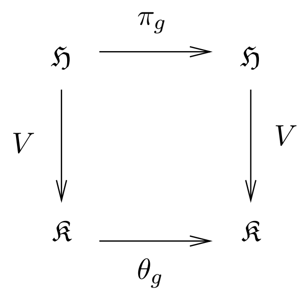
$$
g \to \langle \zeta, \pi_g \xi \rangle \tag{15}
$$

are continuous for all $\zeta, \xi \in \mathfrak{H}$. The theory of unitary representations of locally compact groups, and of Lie groups in particular, is extensive. Here we focus on those (small) parts we need for the development of the theory of the radar ambiguity function.

Two representations π and θ of G on the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ are *equivalent* if there is an isometry $V : \mathfrak{H} \to \mathfrak{K}$ such that

$$V\pi_g = \theta_g V \qquad (g \in G). \tag{16}$$

In pictorial form, π and θ are equivalent if there exists a V such that



commutes. Without the isometry constraint, the operator V is called an *intertwining operator* of π with θ. If every intertwining operator of θ with itself is a scalar multiple of the identity, θ is said to be *irreducible*. This is equivalent to there being no non-trivial subspace of $\mathfrak{K}$ invariant under the action of all of the operators θ_g $(g \in G)$. The connection between invariant subspaces and intertwining operators (via the spectral theorem) which makes these two definitions equivalent is called *Schur's Lemma*.

When the group G is compact every representation is a direct sum of irreducible representations and this decomposition is unique up to equivalence of representations. This is the content of the Peter-Weyl Theorem. Unfortunately, the groups of special interest in radar are not compact and it is not true for these groups that every representation is a direct sum of irreducibles. In any case, every representation is a direct integral of irreducibles; see (Mackey, 1976). In general this decomposition is not unique but, for the groups of major interest to us, uniqueness (almost everywhere) holds.

As we have already seen, it will be necessary to consider multiplier representations as well as ordinary representations. For this, we need a *multiplier*. This is a Borel map $\sigma : G \times G \to \mathbf{T}$, where $\mathbf{T}$ is the group under multiplication of complex numbers of absolute value 1, that satisfies the *cocycle* condition

$$\sigma(g_1, g_2)\sigma(g_1 g_2, g_3) = \sigma(g_1, g_2 g_3)\sigma(g_2, g_3) \qquad (g_1, g_2, g_3 \in G), \tag{17}$$

to which we add the normalization $\sigma(1, 1) = 1$. Then a multiplier representation is a map $\rho : G \to \mathcal{U}(\mathfrak{H})$ that satisfies

$$\rho_{g_1 g_2} = \sigma(g_1, g_2)\rho_{g_1}\rho_{g_2} \qquad (g_1, g_2 \in G), \tag{18}$$

and that is measurable in the same sense that an ordinary representation is continuous. We remark that while the multipliers and multiplier representations considered in this paper are all continuous, it is important for the development of the general theory to permit multipliers and representations that are only Borel. The definitions of equivalence and irreducibility for multiplier representations are unchanged from the ordinary representation case. The *projective unitary group* is

$$\mathcal{P}(\mathfrak{H}) = \mathcal{U}(\mathfrak{H})/\{c\mathbf{I} : c \in \mathbf{T}\}. \tag{19}$$

Multiplier representations give rise to *projective representations* (that is, continuous homomorphisms from G to $\mathcal{P}(\mathfrak{H})$) via composition with the quotient map $p : \mathcal{U}(\mathfrak{H}) \to \mathcal{P}(\mathfrak{H})$. Conversely, since there is a Borel cross section $\eta : \mathcal{P}(\mathfrak{H}) \to \mathcal{U}(\mathfrak{H})$ (that is, a right inverse of p) any projective representation gives rise to a multiplier representation. We note, however, that there are many such cross sections, and so there are many multiplier representations giving rise to the same projective representation — indeed different multipliers are involved in general.

The multipliers on a group G form an abelian group under multiplication. A multiplier (or *cocycle*) σ is a *coboundary* if there is a Borel map $\phi : G \to \mathbf{T}$ such that

$$\sigma(g_1, g_2) = \frac{\phi(g_1)\phi(g_2)}{\phi(g_1 g_2)}. \tag{20}$$

Coboundaries form a subgroup of the group of cocycles of G and two elements of the same coset are said to be *cohomologous.* Cohomologous multipliers have essentially the same representation theory. If σ_1 and σ_2 are cohomologous:

$$\sigma_1(g_1, g_2) = \frac{\phi(g_1)\phi(g_2)}{\phi(g_1 g_2)} \sigma_2(g_1, g_2), \tag{21}$$

and ρ is a σ_1-representation, then a simple check reveals that

$$g \to \phi(g)\rho_g \tag{22}$$

is a σ_2-representation.

As we have stated, a projective representation $\kappa : G \to \mathcal{P}(\mathfrak{H})$ lifts to different multiplier representations using different Borel cross-sections of the quotient map p. However different lifts just produce cohomologous multipliers. If they are then made representations for the same multiplier using the trick described in equation (22), then the two representations are equivalent. Thus the theory of multiplier representations is essentially the theory of projective representations.

One final issue on the general theory of representations needs to be addressed before we return to radar theory. This is that projective/multiplier representations are really ordinary representations but for a different group. Before justifying that we point out that if projective representations did not exist (as the radar ambiguity function demonstrates they should), then we would have to invent them. There is a remarkable and beautiful theory for constructing representations of groups called "the Mackey analysis"; see (Mackey, 1976). To give a detailed exposition of this would take us well beyond the focus of this paper. We merely mention that, in order to use the Mackey analysis to construct *ordinary* representations of a group G, it is necessary to consider projective representations of subgroups of G. On the other hand, if we start trying to construct projective representations of G using the extension of the Mackey analysis to the projective case, then we still only have to consider projective representations of subgroups. Projective representations form a natural completion of the class of ordinary representations.

Now let ρ be a σ-representation of a group G. We show how to make ρ into an ordinary representation of a larger group. We can form a new group $\widetilde{G}$ whose elements are pairs $[g, z]$, where $g \in G$ and $z \in \mathbf{T}$, and with multiplication given by

$$[g_1, z_1][g_2, z_2] = [g_1 g_2, z_1 z_2 \sigma(g_1, g_2)]. \tag{23}$$

The identity element is $(1_G, 1)$ and the inverse is given by

$$[g, z]^{-1} = [g^{-1}, \overline{z\sigma(g, g^{-1})}]. \tag{24}$$

Note that a consequence of the normalization $\sigma(1, 1) = 1$ is that

$$\sigma(g, 1) = \sigma(1, g) = 1 \qquad (g \in G). \tag{25}$$

The group $\widetilde{G}$ is called a *central extension* of the group G by the circle group $\mathbf{T}$. By this we mean that $\mathbf{T}$ (as the subgroup $\{[0, z] : z \in \mathbf{T}\}$) is a subgroup of the centre of $\widetilde{G}$ and the quotient $\widetilde{G}/\mathbf{T}$ is isomorphic to G. Central extensions are classified by the group of cocycles modulo coboundaries. For details we refer the reader to (Maclane, 1975).

Thus, for any σ-representation ρ of G,

$$\pi([g, z]) = \overline{z}\rho_g \tag{26}$$

is an ordinary representation of $\widetilde{G}$. There is an exact correspondence between the σ-representation theory of G and the ordinary representation theory of those representations of $\widetilde{G}$ that restrict on the central subgroup $\mathbf{T}$ to be the homomorphism $[g, z] \to \overline{z}\mathbf{I}$.

4. Representations and Radar

We return to the radar ambiguity function and how representation theory impinges on it. As we have seen in Section 2, the radar ambiguity function is expressible in the form

$$A_{\mathbf{w},\mathbf{v}}(t,f) = \left\langle \mathbf{w}, \rho_{(t,f)}\mathbf{v} \right\rangle, \tag{27}$$

where $(t,f) \to \rho_{(t,f)}$ is a σ-representation of $\mathbf{R}^2$ and

$$\sigma\big((t,f),(t',f')\big) = e^{-2\pi i f' t}. \tag{28}$$

We explore some of the properties of this represention over the next few subsections.

4.1 Basic Properties of the Ambiguity Function

We can quickly deduce four fairly straightforward properties:

Amb1 For all $\mathbf{u},\mathbf{v} \in \mathsf{L}^2(\mathbf{R})$, $A_{\mathbf{u},\mathbf{v}}$ is in $\mathsf{L}^2(\mathbf{R}^2)$ and $\|A_{\mathbf{u},\mathbf{v}}\|_{\mathsf{L}^2(\mathbf{R}^2)} \leq \|\mathbf{u}\|_{\mathsf{L}^2(\mathbf{R})}\|\mathbf{v}\|_{\mathsf{L}^2(\mathbf{R})}$.

Amb2 The map $(\mathbf{u},\mathbf{v}) \to A_{\mathbf{u},\mathbf{v}}$ is conjugate bilinear from $\mathsf{L}^2(\mathbf{R}) \times \mathsf{L}^2(\mathbf{R})$ to $\mathsf{L}^2(\mathbf{R}^2)$.

Amb3 $A_{\mathbf{u},\mathbf{v}}(t,f) = e^{2\pi i f t}\overline{A_{\mathbf{v},\mathbf{u}}(-t,-f)}$.

Amb4 For $\mathbf{u},\mathbf{v} \in \mathsf{L}^2(\mathbf{R})$,

$$A_{\mathcal{F}\mathbf{u},\mathcal{F}\mathbf{v}}(t,f) = e^{2\pi i f t}A_{\mathbf{u},\mathbf{v}}(f,-t). \tag{29}$$

Here, $\mathcal{F}$ denotes the Fourier transform.

Proof: The proof of **Amb1** will be dealt with as part of a more specific theorem in Section 4.3. The conjugate bilinearity is clear, hence **Amb2** follows from **Amb1**. The proof of **Amb3** is a simple calculation. First notice that

$$\rho^*_{(t,f)} = \rho^{-1}_{(t,f)} = e^{2\pi i f t}\rho_{(-t,-f)}, \tag{30}$$

in view of (10). It follows that

$$\begin{aligned} A_{\mathbf{v},\mathbf{u}}(-t,-f) &= \langle \mathbf{v}, \rho_{(-t,-f)}\mathbf{u}\rangle \\ &= \langle \rho^*_{(-t,-f)}\mathbf{v}, \mathbf{u}\rangle \\ &= e^{2\pi i f t}\langle \rho_{(t,f)}\mathbf{v}, \mathbf{u}\rangle \\ &= e^{2\pi i f t}\overline{A_{\mathbf{u},\mathbf{v}}(t,f)}. \end{aligned} \tag{31}$$

To deal with **Amb4**, we note first that, by the Plancherel theorem,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathcal{F}\mathbf{u}, \mathcal{F}\mathbf{v} \rangle \qquad (\mathbf{u}, \mathbf{v} \in \mathsf{L}^2(\mathbf{R})). \tag{32}$$

Further, it is easy to check that

$$\mathcal{F}\left[\rho_{(t,f)}\mathbf{w}\right] = e^{-2\pi i f t}\rho_{(-f,t)}\mathcal{F}[\mathbf{w}]. \tag{33}$$

Now it follows that

$$\begin{aligned} A_{\mathcal{F}\mathbf{u},\mathcal{F}\mathbf{v}}(t,f) &= \left\langle \mathcal{F}\mathbf{u}, \rho_{(t,f)}\mathcal{F}\mathbf{v} \right\rangle \\ &= \left\langle \mathcal{F}\mathbf{u}, e^{-2\pi i f t}\mathcal{F}[\rho_{(f,-t)}\mathbf{v}] \right\rangle \\ &= e^{2\pi i f t}\langle \mathbf{u}, \rho_{(f,-t)}\mathbf{v} \rangle \\ &= e^{2\pi i f t}A_{\mathbf{u},\mathbf{v}}(f,-t). \end{aligned} \tag{34}$$

□

It is remarked that **Amb3** is a particular case of a more general formula applying to any σ-representation ρ:

$$\langle \mathbf{u}, \rho_g \mathbf{v} \rangle = \sigma(g, g^{-1})\overline{\langle \mathbf{v}, \rho_{g^{-1}}\mathbf{u} \rangle} \qquad (g \in G,\ \mathbf{u}, \mathbf{v} \in \mathsf{L}^2(\mathbf{R})). \tag{35}$$

4.2 Irreducibility

The following result is really one of the two most important results in the theory of the radar ambiguity function. The other is Theorem 2.

THEOREM 1 *The σ-representation ρ is irreducible and, moreover, it is the unique irreducible σ-representation of $\mathbf{R}^2$ up to equivalence.*

PROOF: The proof of the uniqueness of ρ would take us too far from the key aim of this paper. It is called the *Stone-von Neumann Theorem* and requires some deep ideas in representations of $\mathbf{R}$. Even the proof of irreducibility relies on some results in harmonic analysis that go well beyond the scope of this paper. To prove irreducibility we use Schur's Lemma. Suppose that $B : \mathsf{L}^2(\mathbf{R}) \to \mathsf{L}^2(\mathbf{R})$ is an intertwining operator of ρ with itself. Then, in particular, it commutes with the translation operators

$$\rho_{(t,0)}(\mathbf{v})(\tau) = \mathbf{v}(\tau - t). \tag{36}$$

Such an object B is known as a *pseudo-measure*. The only fact we need about pseudo-measures is that they have a Fourier transform $\widehat{B}(f)$ such that

$$\widehat{B(\mathbf{v})}(f) = \widehat{B}(f)\widehat{\mathbf{v}}(f) \qquad (f \in \mathbf{R}), \tag{37}$$

and that this Fourier transform specifies a pseudo-measure uniquely. We refer the interested reader to (Katznelson, 1968) for details. Now we note that

$$\widehat{\rho_{(0,f')}(\mathbf{v})}(f) = \widehat{\mathbf{v}}(f + f'). \tag{38}$$

Since B must commute with these operators too, its Fourier transform is translation invariant and so must be constant. It follows that B itself is a scalar multiple of the identity and so ρ is irreducible. □

4.3 Moyal's Identity

Here we establish a result that is one of the key special features of the representation ρ and the ambiguity function. Before we state the result, we establish some terminology and notation. The Hilbert space tensor product of two Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ is the completion of the linear space of all finite formal sums

$$\sum_k \mathbf{u}_k \otimes \mathbf{v}_k, \tag{39}$$

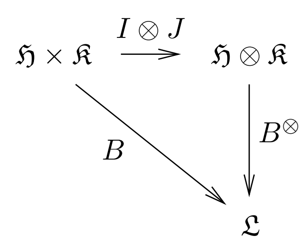
where $\mathbf{u}_k$ and $\mathbf{v}_k$ are in $\mathfrak{H}$ and $\mathfrak{K}$ respectively. The completion is with respect to the norm obtained from the inner product

$$\Big\langle \sum_k \mathbf{u}_k^{(1)} \otimes \mathbf{v}_k^{(1)}, \sum_{k'} \mathbf{u}_{k'}^{(2)} \otimes \mathbf{v}_{k'}^{(2)} \Big\rangle = \sum_{k,k'} \big\langle \mathbf{u}_k^{(1)}, \mathbf{u}_{k'}^{(2)} \big\rangle_{\mathfrak{H}} \big\langle \mathbf{v}_k^{(1)}, \mathbf{v}_{k'}^{(2)} \big\rangle_{\mathfrak{H}}. \tag{40}$$

We assume there is a conjugate linear isometry $J : \mathfrak{K} \to \mathfrak{K}$ (in fact this will be conjugation in $\mathsf{L}^2(\mathbf{R})$ in our context). Evidently, given any conjugate bilinear map $B : \mathfrak{H} \times \mathfrak{K} \to \mathfrak{L}$ that satisfies

$$\Big\| \sum_k B(\mathbf{u}_k, \mathbf{v}_k) \Big\|_{\mathfrak{L}} \leq \Big\| \sum \mathbf{u}_k \otimes \mathbf{v}_k \Big\|_{\mathfrak{H} \otimes \mathfrak{K}} \tag{41}$$

for any finite set $\{(\mathbf{u}_k, \mathbf{v}_k)\}$ of elements of $\mathfrak{H} \times \mathfrak{K}$, there is a continuous linear map $B^{\otimes} : \mathfrak{H} \otimes \mathfrak{K} \to \mathfrak{L}$ such that the diagram



extends to a continuous linear map from $\mathfrak{H} \otimes \mathfrak{K} \to \mathfrak{L}$. We note that the tensor product of $\mathsf{L}^2(\mathbf{R})$ with itself is just $\mathsf{L}^2(\mathbf{R}^2)$. The following theorem is in essence Moyal's identity.

Theorem 2 *The map* $(\mathbf{u}, J\mathbf{v}) \to A_{\mathbf{u},\mathbf{v}}$ *extends to an isometry from* $\mathsf{L}^2(\mathbf{R}^2)$ *to itself.*

Proof: It is easy to see that this map (for finite sums of simple tensors in the first instance) is just

$$\Phi(F)(t, f) = \int F(\tau, \tau - t)e^{2\pi i f \tau}\, d\tau, \tag{42}$$

which, since we know that the Hilbert space tensor product of $\mathsf{L}^2(\mathbf{R})$ with itself is just $\mathsf{L}^2(\mathbf{R}^2)$, clearly satisfies the condition (41). If we write $W : \mathsf{L}^2(\mathbf{R}^2) \to \mathsf{L}^2(\mathbf{R}^2)$ for the map

$$W(F)(\tau, t) = F(\tau, \tau - t), \tag{43}$$

it is clear that it is an isometry of $\mathsf{L}^2(\mathbf{R}^2)$ and that Φ is just $(I \otimes \mathcal{F}) \circ W$ where $\mathcal{F}$ is the one dimensional Fourier transform. The result is now clear. □

Corollary 3 *For* $\mathbf{u}_1, \mathbf{v}_1, \mathbf{u}_2, \mathbf{v}_2 \in \mathsf{L}^2(\mathbf{R})$,

$$\langle A_{\mathbf{u}_1,\mathbf{v}_1}, A_{\mathbf{u}_2,\mathbf{v}_2}\rangle_{\mathsf{L}^2(\mathbf{R}^2)} = \langle \mathbf{u}_1, \mathbf{u}_2\rangle_{\mathsf{L}^2(\mathbf{R})}\langle \mathbf{v}_1, \mathbf{v}_2\rangle_{\mathsf{L}^2(\mathbf{R})}. \tag{44}$$

It is equation (44) that is commonly referred to as *Moyal's Identity*. This has in turn another corollary.

Corollary 4 *For* $\mathbf{u}, \mathbf{v} \in \mathsf{L}^2(\mathbf{R})$,

$$\|A_{\mathbf{v},\mathbf{v}}\|_{\mathsf{L}^2(\mathbf{R}^2)} = \|\mathbf{v}\|^2. \tag{45}$$

This is a form of the *Heisenberg uncertainty principle*. To explain its significance, we need to make a few remarks about the way radars normally operate. When the return comes back into the receiver from a scene, noise is added by the receiver. This is just thermal noise in the electronic components. To a very good approximation this is *white Gaussian noise* — a stationary Gaussian random process in which restrictions to disjoint intervals are independent. Such a process (with finite energy) cannot exist mathematically, but as already stated, this is an approximation. If one asks what waveform $\mathbf{v}$ when used for filtering in the receiver will maximize the signal-to-noise ratio, one can show that the correct choice is $\mathbf{v} = \mathbf{w}$. So it is normal (at least in theoretical discussions of radar) to assume that $\mathbf{v} = \mathbf{w}$ — the so-called *matched filter*.

We note that by (3) the processed return is just the convolution of the range-Doppler scene with the ambiguity. Evidently then, so that we can extract the scene from the return, we would like $A_{\mathbf{v},\mathbf{v}}$ (or $A_{\mathbf{v},\mathbf{w}}$) to be a delta function or close to it. Moyal's identity makes this impossible. Clearly the value of $A_{\mathbf{v},\mathbf{v}}$ at the origin is just $\|\mathbf{v}\|^2$ and this is also its L^2 norm, so that it must have considerable spread.

As we shall see later, an important collection of waveforms (in theory but not in practice) are the Hermite functions. In fact it will be convenient for us to use a renormalized form of them. We write

$$\mathbf{v}_n(t) = \pi^{-1/4}(2^n n!)^{-1/2}(-1)^n e^{-t^2/2} H_n(t), \tag{46}$$

where H_n is the nth Hermite polynomial, defined implicitly by the generating equation

$$\sum_n \frac{1}{n!} H_n(t) z^n = e^{-z^2+2zt}. \tag{47}$$

The $\mathbf{v}_n$ form an orthonormal basis of $\mathsf{L}^2(\mathbf{R})$. It follows from Moyal's identity that the collection $\{A_{\mathbf{v}_n,\mathbf{v}_m}\}_{n,m}$ therefore forms an orthonormal basis of $\mathsf{L}^2(\mathbf{R}^2)$.

4.4 The Symmetric Ambiguity Function

In practical radar systems, the location (t, f) of a target is usually estimated by maximising the magnitude of the inner product between the received waveform $\mathbf{v}(\tau)$ and the adjusted version $\rho_{(t,f)}(\mathbf{w})(\tau)$ of the transmitted waveform $\mathbf{w}(\tau)$. Here, $\rho_{(t,f)}(\mathbf{w})(\tau)$ corresponds to what would have been received in the ideal case had there been a single target at "distance" t moving at "velocity" f. Therefore, it is only the magnitude of the ambiguity function, and not its phase, which provides useful information about the performance of the system.

Let $\phi : \mathbf{R}^2 \to \mathbf{T}$ be an arbitrary Borel map. Then the magnitude of $\langle \mathbf{v}, \phi(t,f)\rho_{(t,f)}\mathbf{w} \rangle$ equals the magnitude of the ambiguity function $A_{\mathbf{v},\mathbf{w}}(t, f)$. Moreover, the results of Section 3 imply that $\rho^{(\nu)}_{(t,f)} = \phi(t, f)\rho_{(t,f)}$ is actually a ν-representation of $\mathbf{R}^2$ for some multiplier ν cohomologous to σ. Therefore, what is of interest is not the properties of the particular ambiguity function $A_{\mathbf{v},\mathbf{w}}(t, f)$, but rather the properties shared by all functions of the form $\langle \mathbf{v}, \rho^{(\nu)}_{(t,f)}\mathbf{w} \rangle$. Since multiplier representation theory does not distinguish between cohomologous multipliers, this shows that *radar theory really is concerned with the σ-representation theory of $\mathbf{R}^2$, no more and no less.*

Replacing σ by a cohomologous multiplier ν changes (the phase but not the magnitude of) the corresponding ambiguity function. Being able to work with different ambiguity functions can simplify calculations. The symmetric ambiguity function is now derived by considering the multiplier ν defined by

$$\nu\big((t,f),(t',f')\big) = e^{-\pi i(f't-t'f)}. \tag{48}$$

This is cohomologous to σ because

$$\frac{\nu((t,f),(t',f'))}{\sigma((t,f),(t',f'))} = e^{\pi i(t'f+f't)} = \frac{e^{-\pi itf}e^{-\pi it'f'}}{e^{-\pi i(t+t')(f+f')}}. \tag{49}$$

In fact, one can show that up to coboundaries and automorphisms of $\mathbf{R}^2$, this is the only cocycle on $\mathbf{R}^2$. Switching to this multiplier from σ, we obtain an alternative formula for ρ, this time as a ν-representation. From the formulae (22) and (49), we obtain

$$\rho^{(\nu)}_{(t,f)}(\mathbf{v})(\tau) = e^{\pi ift}\rho_{(t,f)}(\mathbf{v})(\tau) = e^{\pi if(t-2\tau)}\mathbf{v}(\tau-t), \qquad (t,f)\in\mathbf{R}^2. \tag{50}$$

This results in the following form for the radar ambiguity function

$$A^{(\nu)}_{\mathbf{u},\mathbf{v}}(t,f) = \int_{\mathbf{R}} \mathbf{u}(\tau+\frac{t}{2})\overline{\mathbf{v}(\tau-\frac{t}{2})}e^{2\pi if\tau}\,d\tau. \tag{51}$$

This is referred to as the *symmetric* form of the ambiguity function.

Note that because $\nu((t,f),(-t,-f)) = 1$, we obtain from (35) the tidier form

$$A^{(\nu)}_{\mathbf{u},\mathbf{v}}(t,f) = \overline{A_{\mathbf{v},\mathbf{u}}(-t,-f)} \tag{52}$$

of **Amb3**. In a similar vein, the new form of **Amb4** is

$$A^{(\nu)}_{\mathcal{F}\mathbf{u},\mathcal{F}\mathbf{v}}(t,f) = A^{(\nu)}_{\mathbf{u},\mathbf{v}}(f,-t). \tag{53}$$

4.5 The Heisenberg Group

Using the ideas of Section 3 and the cocycle σ corresponding to ρ, it is possible to form a central extension of $\mathbf{R}^2$ by $\mathbf{T}$. The result is a group G_0 whose centre Z_0 is (in this case) isomorphic to $\mathbf{T}$ and for which G_0/Z_0 is isomorphic to $\mathbf{R}^2$. It is customary instead to use the $\mathbf{R}$-valued cocycle $\sigma_{\mathbf{R}}$ given by

$$\sigma_{\mathbf{R}}\big((t,f),(t',f')\big) = f't, \tag{54}$$

so that $\sigma = e^{-2\pi i\sigma_{\mathbf{R}}}$. The result of applying the trick of equation (23) in this context produces a central extension of $\mathbf{R}^2$ by $\mathbf{R}$, which is called the (3-dimensional) *Heisenberg* group

$$\mathcal{H} = \begin{pmatrix} 1 & t & z \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}. \tag{55}$$

The corresponding representation of $\mathcal{H}$ is

$$\widetilde{\rho}(t, f, z)(\mathbf{v})(\tau) = e^{2\pi iz} e^{-2\pi i f\tau} \mathbf{v}(\tau - t). \tag{56}$$

There is another extension (but isomorphic *qua* central extensions) formed using ν, but we refrain from giving the details. In fact, we continue to present the theory of the radar ambiguity function in terms of multiplier representations of $\mathbf{R}^2$ rather than the slightly more customary ordinary representation theory of the Heisenberg group.

4.6 The Bargmann-Segal Representation

While the Stone-von Neumann Theorem tells us that there is only one irreducible σ-representation of $\mathbf{R}^2$, or equivalently that there is only one irreducible representation of the Heisenberg group that restricts to the homomorphism $z \to e^{2\pi iz}$ in the centre of $\mathcal{H}$, there are many equivalent ways of seeing this representation. The way we have discussed so far is called the *Schroedinger representation* of $\mathcal{H}$. Here and in the following section we present two other ways of looking at this representation.

It is shown in (Wilcox, 1991) that the Hermite waveforms $\mathbf{v}_n(t)$, defined in (46), have ambiguity functions whose peak at the origin is as sharp as possible (in a sense made precise there). The Hermite waveforms are also the eigenfunctions of a particular linear operator defined in Section 4.8. Since the Hermites appear to play a fundamental role, it is desirable to construct a σ-representation on a new Hilbert space $\mathfrak{F}$ such that the waveforms corresponding to the Hermites are as simple as possible. Such a representation is stated below.

Consider the space $\mathfrak{F}$ of entire functions $\mathbf{a}(z) = \sum_n a_n z^n$ satisfying $\sum_n n!\, |a_n|^2 < \infty$. This is a Hilbert space with the inner product

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_n n!\, a_n \overline{b_n}. \tag{57}$$

This is actually a *reproducing kernel* Hilbert space, in that there exist elements $\mathbf{e}_u$, for any $u \in \mathbf{C}$, such that

$$\langle \mathbf{a}, \mathbf{e}_u \rangle = \mathbf{a}(u), \qquad \mathbf{a} \in \mathfrak{F}. \tag{58}$$

Furthermore, the vectors

$$\mathbf{j}_n(z) = \frac{z^n}{\sqrt{n!}}, \qquad n = 0, 1, \cdots \tag{59}$$

form an orthonormal basis for $\mathfrak{F}$.

We can explicitly write down an isometry $\Psi : \mathsf{L}^2(\mathbf{R}) \to \mathfrak{F}$ which maps $\mathbf{v}_n(t)$ to $\mathbf{j}_n(z)$, namely

$$\begin{aligned} \Psi(\mathbf{u})(z) &= \sum_{n=0}^{\infty} \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{j}_n(z) \\ &= \int_{\mathbf{R}} \frac{1}{\pi^{1/4}} e^{-(z^2+t^2)/2-\sqrt{2}zt} u(t)\, dt. \end{aligned} \tag{60}$$

Here, the second line is obtained with the help of (47).

The isometry Ψ maps the representation ρ to the equivalent representation $\rho^{\mathrm{BS}} = \Psi \circ \rho \circ \Psi^{-1}$ on $\mathfrak{F}$. This is called the *Bargmann-Segal σ-representation* of $\mathbf{R}^2$. It is given explicitly by

$$\rho^{\mathrm{BS}}_{(t,f)}(\mathbf{a})(z) = e^{-\frac{t^2+4\pi^2 f^2}{4} + \frac{z}{\sqrt{2}}(2\pi i f - t) - \pi i f t}\, \mathbf{a}\left(z + \frac{t + 2\pi i f}{\sqrt{2}} \right). \tag{61}$$

The derivation of (61) is postponed until Section 4.8.

4.7 The Lattice Representation

The third equivalent version of the representation ρ is the so-called *lattice representation.* It arises as an induced σ-representation from the lattice subgroup $\mathbf{Z}^2$ of $\mathbf{R}^2$ or equivalently as an induced ordinary representation from the corresponding subgroup of the Heisenberg group $\mathcal{H}$. The Hilbert space here is the space of all functions $\mathbf{r} : \mathbf{R}^2 \to \mathbf{C}$ satisfying

$$\mathbf{r}(x+n, y+m) = e^{-2\pi i n y} \mathbf{r}(x, y), \tag{62}$$

and which are square integrable in the sense that

$$\int_0^1 \int_0^1 |\mathbf{r}(x,y)|^2 \, dx \, dy < \infty. \tag{63}$$

Note that, in view of (62), the integrand is periodic. This space has the obvious inner product. Now we can define a σ-representation ρ^{L} of $\mathbf{R}^2$ by

$$\rho^{\mathrm{L}}_{(t,f)}(\mathbf{r})(x,y) = e^{2\pi i f x} \mathbf{r}(x+t, y+f). \tag{64}$$

It is not hard to see that this is a σ-representation. To prove irreducibility we show directly that it is equivalent to ρ. The intertwining operator is known as the *Weil-Brezin-Zaks* transform and is given by

$$Z[\mathbf{u}](x,y) = \sum_{k=-\infty}^{\infty} e^{-2\pi i k y} \mathbf{u}(k - x) \qquad (\mathbf{u} \in \mathsf{L}^2(\mathbf{R})). \tag{65}$$

It is straightforward to check $Z[\mathbf{u}]$ satisfies both (62) and (63). Its inverse is given by

$$Z^{-1}[\mathbf{r}](\tau) = \int_0^1 \mathbf{r}(-\tau, y)\, dy. \tag{66}$$

To check the intertwining property, we observe

$$\begin{aligned} Z[\rho_{(t,f)}(\mathbf{u})](x,y) &= e^{2\pi i f x} \sum_{k=-\infty}^{\infty} e^{-2\pi i k(f+y)} \mathbf{u}(k - x - t) \\ &= \rho^{\mathrm{L}}_{(t,f)}(Z[\mathbf{u}])(x,y). \end{aligned} \tag{67}$$

Note that this forces Z to be a scalar multiple of an isomorphism and it is straightforward to see from this that it is indeed an isomorphism.

Although not discussed here, there are links between the lattice representation and number theory. For instance, the Jacobi theta function appears in the expression for $Z[\mathbf{v}_0](x,y)$, where $\mathbf{v}_0$ is the zeroth Hermite waveform defined in (46).

4.8 The Lie Algebra Representation

Recall that the σ-representation ρ extends to a representation on the Heisenberg group $\mathcal{H}$. Since $\mathcal{H}$ is a Lie group, the representation ρ induces a Lie algebra representation on the Lie algebra $\mathfrak{h}$ associated with $\mathcal{H}$. The Hermite polynomials naturally arise in this framework as the eigenvectors of a particular linear operator.

The Heisenberg group has a Lie algebra $\mathfrak{h}$ which, as an additive group, is $\mathbf{R}^3$ and has generators T, F, Z satisfying

$$[T, F] = Z, \quad [T, Z] = 0, \quad [F, Z] = 0. \tag{68}$$

The representation $\widetilde{\rho}$ of $\mathcal{H}$ produces a representation, which we also denote by $\widetilde{\rho}$, of the Lie algebra by unbounded operators on $\mathsf{L}^2(\mathbf{R})$. In the case of $\widetilde{\rho}$, the representations of the generators are

$$\widetilde{\rho}(F) = -2\pi i t; \quad \widetilde{\rho}(T) = -\frac{d}{dt}; \quad \widetilde{\rho}(Z) = 2\pi i. \tag{69}$$

It is easily checked that these satisfy (68). Let

$$A = \frac{1}{\sqrt{2}}\big(T + \frac{1}{2\pi i} F\big) \tag{70}$$

and observe that

$$\widetilde{\rho}(A) = \frac{-1}{\sqrt{2}}\big(\frac{d}{dt} + t\big). \tag{71}$$

Its adjoint is

$$\widetilde{\rho}(A)^* = \frac{1}{\sqrt{2}}\big(\frac{d}{dt} - t\big). \tag{72}$$

We define $N = \widetilde{\rho}(A)^*\widetilde{\rho}(A)$ and remark that it is a self-adjoint operator.

The normalized solution of $\widetilde{\rho}(A)(\mathbf{v}) = 0$ in $\mathsf{L}^2(\mathbf{R})$ is

$$\mathbf{v}_0(t) = \frac{1}{\pi^{1/4}} e^{-t^2/2}, \tag{73}$$

the zeroth Hermite function in (46). In fact, the eigenvalues of N are $\{0,\ 1,\ 2,\ \ldots\}$ and the corresponding eigenvectors of N are the $\mathbf{v}_n$ as defined in (46). Indeed, one can show easily that

$$\begin{aligned} \widetilde{\rho}(A)^*(\mathbf{v}_n) &= \sqrt{n+1}\,\mathbf{v}_{n+1} \\ \widetilde{\rho}(A)(\mathbf{v}_n) &= \sqrt{n}\,\mathbf{v}_{n-1}. \end{aligned} \tag{74}$$

The Bargmann-Segal representation stated in Section 4.6 is now derived. Define the Hilbert space $\mathfrak{F}$ as in Section 4.6 and let Ψ be the isometry in (60). Deriving an expression for $\rho^{\mathrm{BS}} = \Psi \circ \rho \circ \Psi^{-1}$ directly is difficult, but it can be found indirectly by first calculating the corresponding Lie algebra representation $\widetilde{\rho^{\mathrm{BS}}}$, as follows.

Since $\rho = \Psi^{-1} \circ \rho^{\mathrm{BS}} \circ \Psi$, it follows that $\widetilde{\rho} = \Psi^{-1} \circ \widetilde{\rho^{\mathrm{BS}}} \circ \Psi$ and $\widetilde{\rho}^* = \Psi^{-1} \circ \widetilde{\rho^{\mathrm{BS}}}^* \circ \Psi$. Therefore, $\widetilde{\rho}(A)(\mathbf{v}_0) = 0$ implies $\widetilde{\rho^{\mathrm{BS}}}(A)(\mathbf{j}_0) = 0$, that is, $\widetilde{\rho^{\mathrm{BS}}}(A)(1) = 0$. Similarly, $\widetilde{\rho}(A)(\mathbf{v}_n) = \sqrt{n}\,\mathbf{v}_{n-1}$ implies $\widetilde{\rho^{\mathrm{BS}}}(A)(z^n) = nz^{n-1}$. Therefore,

$$\widetilde{\rho^{\mathrm{BS}}}(A) = \frac{d}{dz}, \tag{75}$$

or upon substituting for A,

$$\widetilde{\rho^{\mathrm{BS}}}(T) + \frac{1}{2\pi i}\widetilde{\rho^{\mathrm{BS}}}(F) = \sqrt{2}\frac{d}{dz}. \tag{76}$$

It is readily shown that $\widetilde{\rho}(T)$ and $\widetilde{\rho}(F)$ are skew-adjoint, hence so are $\widetilde{\rho^{\mathrm{BS}}}(T)$ and $\widetilde{\rho^{\mathrm{BS}}}(F)$. That is to say, $\widetilde{\rho^{\mathrm{BS}}}(T)^* = -\widetilde{\rho^{\mathrm{BS}}}(T)$ and similarly for $\widetilde{\rho^{\mathrm{BS}}}(F)$. Therefore, taking the adjoint of (76) yields the new equation

$$-\widetilde{\rho^{\mathrm{BS}}}(T) + \frac{1}{2\pi i}\widetilde{\rho^{\mathrm{BS}}}(F) = \sqrt{2}z \tag{77}$$

where use has been made of the facts that the adjoint of $\frac{d}{dz}$ is z and the adjoint of a complex number is its complex conjugate. Solving the equations (76) and (77) gives

$$\widetilde{\rho^{\mathrm{BS}}}(T) = \frac{1}{\sqrt{2}}\left(\frac{d}{dz} - z\right), \quad \widetilde{\rho^{\mathrm{BS}}}(F) = \sqrt{2}\pi i\left(\frac{d}{dz} + z\right). \tag{78}$$

Exponentiating these two operators shows that

$$\begin{aligned}\rho^{\mathrm{BS}}_{(t,0)}(\mathbf{a})(z) &= \exp\left(-\frac{t^2}{4} - \frac{tz}{\sqrt{2}}\right)\mathbf{a}\left(z + \frac{t}{\sqrt{2}}\right), \\ \rho^{\mathrm{BS}}_{(0,f)}(\mathbf{a})(z) &= \exp\left(-\pi^2 f^2 + \sqrt{2}\pi i f z\right)\mathbf{a}\left(z + \sqrt{2}\pi i f\right).\end{aligned} \tag{79}$$

Equation (61) now follows upon noting that $\rho^{\mathrm{BS}}_{(t,f)} = \rho^{\mathrm{BS}}_{(0,f)}\rho^{\mathrm{BS}}_{(t,0)}$.

4.9 The Metaplectic Representation

Here we consider automorphisms of $\mathbf{R}^2$ that preserve the structure we have discussed so far. In fact, it is more customary at this point to work with the equivalent multiplier ν, defined in (48), rather than the multiplier σ. We consider, then, the continuous automorphisms α of the group $\mathbf{R}^2$ that preserve the multiplier ν in the sense that

$$\nu(\alpha(t,f), \alpha(t',f')) = \nu((t,f),(t',f')), \qquad (t,f),(t',f') \in \mathbf{R}^2. \tag{80}$$

These are just the members of $\mathsf{SL}(2,\mathbf{R})$ — 2×2 matrices with determinant 1. Evidently, for any such automorphism, $\rho^{(\nu)} \circ \alpha$ is an irreducible ν-representation of $\mathbf{R}^2$ and so, up to equivalence, must be $\rho^{(\nu)}$ itself by the Stone-von Neumann Theorem. Thus there exists a unitary operator $U(\alpha)$ such that

$$U(\alpha)^{-1}\rho^{(\nu)}_{(t,f)}U(\alpha) = \rho^{(\nu)}_{\alpha(t,f)}, \qquad (t,f) \in \mathbf{R}^2. \tag{81}$$

Since $\rho^{(\nu)}$ is irreducible, $U(\alpha)$ is unique up to a scalar multiple.

Moreover, $U(\alpha\beta)$ must have the same effect on $\rho^{(\nu)}$ as $U(\alpha)U(\beta)$ and so, by the irreducibility of $\rho^{(\nu)}$, these two must also differ by a multiplicative constant. In other words, the map $\alpha \to U(\alpha)$ is a projective representation of $\mathsf{SL}(2,\mathbf{R})$. In fact, the multiplier is cohomologous to a two-valued one. Thus, there is a double covering $\widetilde{\mathsf{SL}(2,\mathbf{R})}$ of $\mathsf{SL}(2,\mathbf{R})$ and U lifts to a unitary representation of $\widetilde{\mathsf{SL}(2,\mathbf{R})}$ on $\mathsf{L}^2(\mathbf{R})$. This is called the *metaplectic representation* of $\widetilde{\mathsf{SL}(2,\mathbf{R})}$. (The representation is not irreducible, but rather, $\mathsf{L}^2(\mathbf{R})$ decomposes as the direct sum of two subspaces, where the representation restricted to either of the two subspaces is irreducible.)

5. Ambiguity Functions

This section uses the earlier results on the σ-representation theory of $\mathbf{R}^2$ to establish several facts about ambiguity functions.

5.1 Ambiguity Functions of the Hermites

Let $\rho_{nm}(t,f) = \langle \mathbf{v}_n, \rho_{(t,f)}\mathbf{v}_m \rangle$ denote the ambiguity functions associated with the Hermite waveforms $\mathbf{v}_n$ defined in (46). This section outlines how a closed form expression for $\rho_{nm}(t,f)$ can be obtained.

The Hermite waveforms are simpler in the Bargmann-Segal space, so the first step is to notice that $\rho_{nm}(t,f) = \langle \mathbf{j}_n, \rho^{\mathrm{BS}}_{(t,f)}\mathbf{j}_m \rangle$ because the isometry Ψ maps $\mathbf{v}_n$ to $\mathbf{j}_n$; see Section 4.6 for notation. Next, write down the generating function

$$\begin{aligned} G(t,f;a,b) &= \left\langle \mathbf{e}_{\bar{a}}, \rho^{\mathrm{BS}}_{(t,f)}(\mathbf{e}_b) \right\rangle \\ &= \sum_{n,m=0}^{\infty} \left\langle \mathbf{j}_n, \rho^{\mathrm{BS}}_{(t,f)}(\mathbf{j}_m) \right\rangle \frac{a^n b^m}{\sqrt{n!\,m!}} \end{aligned} \tag{82}$$

where the second equality follows from the fact that $\mathbf{e}_u$, defined implicitly by (58), is given explicitly by

$$\mathbf{e}_u = \sum_{k=0}^{\infty} \frac{\bar{u}^k}{\sqrt{k!}} \mathbf{j}_k. \tag{83}$$

Using the reproducing kernel property (58) shows that

$$\begin{aligned} G(t,f;a,b) &= \overline{\rho^{\mathrm{BS}}_{(t,f)}(\mathbf{e}_b)(\bar{a})} \\ &= e^{-\frac{t^2+4\pi^2 f^2}{4} - \frac{a}{\sqrt{2}}(t+2\pi i f) + \pi i f t} e^{b\left(a + (t-2\pi i f)/\sqrt{2}\right)}. \end{aligned} \tag{84}$$

Equating coefficients of $a^n b^m$ in (82) and (84) results in an explicit expression for $\rho_{nm}(t,f) = \left\langle \mathbf{j}_n, \rho^{\mathrm{BS}}_{(t,f)}\mathbf{j}_m \right\rangle$. Moreover, making the substitution $(t,f) = (r\cos\theta, r\sin\theta)$ allows this expression to be written in terms of the Laguerre polynomials. See (Wilcox, 1991) or (Miller, 1991) for details.

5.2 Symmetries of Ambiguity Functions

This section introduces the basic machinery for studying the possible symmetries of an ambiguity function. Although only the magnitude $|A(t,f)|$ of an ambiguity function $A(t,f)$ is of interest in general, it is significantly simpler to study symmetries of $A(t,f)$ rather than of $|A(t,f)|$. Changing multipliers changes the phase of the ambiguity function, thereby potentially altering the symmetry. It is therefore important to note that this section chooses to work with the symmetric ambiguity function defined in Section 4.4.

Recall from Section 4.9 that the elements of $\mathsf{SL}(2,\mathbf{R})$ preserve the multiplier ν. Moreover, from (81), it follows that if $A^{(\nu)}(t,f)$ is an ambiguity function then so too is $A^{(\nu)}\big(\alpha(t,f)\big)$ for any $\alpha \in \mathsf{SL}(2,\mathbf{R})$. Indeed,

$$\begin{aligned} A^{(\nu)}_{\mathbf{u},\mathbf{v}}\big(\alpha(t,f)\big) &= \big\langle \mathbf{u}, \rho^{(\nu)}_{\alpha(t,f)}\mathbf{v}\big\rangle \\ &= \big\langle \mathbf{u}, U(\alpha)^{-1}\rho^{(\nu)}_{(t,f)}U(\alpha)\mathbf{v}\big\rangle \\ &= A^{(\nu)}_{U(\alpha)\mathbf{u},U(\alpha)\mathbf{v}}(t,f). \end{aligned} \tag{85}$$

It follows that if $\mathcal{S}$ is a subgroup of $\mathsf{SL}(2,\mathbf{R})$ and the waveform $\mathbf{u}$ satisfies $U(\alpha)\mathbf{u} = \mathbf{u}$ for all $\alpha \in \mathcal{S}$ then the ambiguity function $A^{(\nu)}_{\mathbf{u},\mathbf{u}}$ is symmetric with respect to $\mathcal{S}$. In fact, since it is shown in (Miller, 1991) that $A^{(\nu)}_{\mathbf{u},\mathbf{u}} = A^{(\nu)}_{\mathbf{u}',\mathbf{u}'}$ if and only if $\mathbf{u} = \lambda\mathbf{u}'$ for some $\lambda \in \mathbf{T}$, we say that $\mathbf{u}$ has an $\mathcal{S}$-symmetric ambiguity function if and only if there exists a function $\lambda : \mathcal{S} \to \mathbf{T}$ such that $U(\alpha)\mathbf{u} = \lambda(\alpha)\mathbf{u}$ for all $\alpha \in \mathcal{S}$.

One interesting subgroup of $\mathsf{SL}(2,\mathbf{R})$ is the rotation group. However, it is more natural to rotate a dilated version of the ambiguity function so that the units of time and frequency are compatible. Let $\mathcal{S}$ denote the group whose elements $S(\theta)$ are dilated rotations, namely

$$\begin{aligned} S(\theta) &= \begin{pmatrix} \sqrt{2\pi} & 0 \\ 0 & \frac{1}{\sqrt{2\pi}} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2\pi}} & 0 \\ 0 & \sqrt{2\pi} \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -2\pi\sin\theta \\ \frac{1}{2\pi}\sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in [0,2\pi). \end{aligned} \tag{86}$$

Note that $\mathcal{S}$ is a subgroup of $\mathsf{SL}(2,\mathbf{R})$ and that $S(-\theta)$ is the inverse of $S(\theta)$. It is now shown that the only waveforms with rotationally symmetric (that is, $\mathcal{S}$-symmetric) ambiguity functions are the Hermites.

Anticipating the involvement of the Hermites, we choose to work in Bargmann-Segal space. Define $U^{\mathrm{BS}}(\alpha) = \Psi \circ U(\alpha) \circ \Psi^{-1}$ so that $\rho^{\mathrm{BS}(\nu)}_{\alpha(t,f)} = U^{\mathrm{BS}}(\alpha)^{-1}\rho^{\mathrm{BS}(\nu)}_{(t,f)}U^{\mathrm{BS}}(\alpha)$. Here, Ψ is the isomorphism defined in (60) and $\rho^{\mathrm{BS}(\nu)}$ is the ν-representation analogue of (61), namely

$$\rho^{\mathrm{BS}(\nu)}_{(t,f)}(\mathbf{a})(z) = e^{-\frac{t^2+4\pi^2f^2}{4} + \frac{z}{\sqrt{2}}(2\pi i f - t)}\,\mathbf{a}\Big(z + \frac{t+2\pi i f}{\sqrt{2}}\Big). \tag{87}$$

It is claimed

$$U^{\mathrm{BS}}\big(S(\theta)\big)(\mathbf{a})(z) = \mathbf{a}(e^{i\theta}z). \tag{88}$$

Indeed, $S(\theta)(t,f) = \big(t\cos\theta - 2\pi f\sin\theta, (2\pi)^{-1}t\sin\theta + f\cos\theta\big)$, hence

$$\begin{aligned} \rho^{\mathrm{BS}(\nu)}_{S(\theta)(t,f)}(\mathbf{a})(z) &= e^{-\frac{t^2+4\pi^2f^2}{4} + \frac{z}{\sqrt{2}}(2\pi i f - t)e^{-i\theta}}\,\mathbf{a}\Big(z + \frac{t+2\pi i f}{\sqrt{2}}e^{i\theta}\Big) \\ &= U^{\mathrm{BS}}\big(S(\theta)\big)^{-1}\rho^{\mathrm{BS}(\nu)}_{(t,f)}U^{\mathrm{BS}}\big(S(\theta)\big)(\mathbf{a})(z), \end{aligned} \tag{89}$$

as required. The rotationally invariant waveforms in the Bargmann-Segal domain are thus those waveforms $\mathbf{a}(z)$ which satisfy

$$\forall\theta \in [0, 2\pi), \quad \mathbf{a}(e^{i\theta}z) = \lambda(\theta)\mathbf{a}(z) \tag{90}$$

for some λ mapping θ into $\mathbf{T}$. The only solutions are the Hermite waveforms $\mathbf{a}(z) = \mathbf{j}_n(z)$, and scalar multiples of them, for $n = 0, 1, \cdots$.

It is remarked that if the ordinary rotations $R(\theta)$ are considered instead of the dilated rotations $S(\theta)$, then $U\big(R(\theta)\big)$ turns out to be the fractional Fourier transform. Indeed, we know from (53) that $U\big(R(\theta)\big)$ is the Fourier transform if $\theta = -\pi/2$. Thus, $U\big(R(\theta)\big)$ embeds the Fourier transform in a one parameter group.

5.3 Multiple Waveforms

This section presents an intriguing observation about how an ambiguity function hypothetically can be made to resemble the ideal delta function, thus overcoming the Heisenberg uncertainty principle.

Assume it was somehow possible to transmit simultaneously but separately many different waveforms $\{\mathbf{u}_n\}_{n=1}^N$ and to receive the returns separately too. What we have in mind is not using different portions of the electromagnetic spectrum, which does not separate the waveforms in the strict sense we require, but rather to imagine the hypothetical situation of multiple universes where the target exists in each one but a different waveform can be used to detect it in each universe.

If the waveforms are orthogonal and have equal energy with total energy one, so that $\|\mathbf{u}_n\|^2 = N^{-1}$, then it follows from Moyal's identity that the ambiguity functions $A_{\mathbf{u}_n,\mathbf{u}_n}(t, f)$ are orthogonal, and in particular, the ambiguity function $A(t, f) = \sum_{n=1}^N A_{\mathbf{u}_n,\mathbf{u}_n}(t, f)$ of the whole system has norm

$$\|A(t, f)\|^2 = \sum_{n=1}^{N} \|A_{\mathbf{u}_n,\mathbf{u}_n}\|^2 = \sum_{n=1}^{N} \|\mathbf{u}_n\|^4 = \frac{1}{N}. \tag{91}$$

This shows that there is less volume under the ambiguity surface for a given total energy as N increases. Therefore, since $A(0, 0) = 1$, the total ambiguity function can be made to approach the ideal delta function as N approaches infinity.

6. The Wide Band Case

Our description of the Doppler effect is actually only an approximation that works when the radial velocities of moving objects in the scene are much smaller than the speed of light, and the transmitted signal has a spectrum in a narrow band around its carrier frequency. This is, of course, a valid assumption on most circumstances in radar. However, in sonar, which works on similar principles, it can often be the case that objects move at a non-trivial proportion of the speed of sound in water (around 1500 metres per second). Here it is appropriate to replace the "narrow band approximation" by the so-called "wide band theory". As the name implies, this theory is also appropriate when the spectrum of the transmit signal is broad, as is the case, for example, where the transmit signal is a very short pulse. Such signals are not typical in conventional radar systems, but again are in sonar, where the pulse is often created by a small explosive charge.

The effect of Doppler is not, as we have suggested in (2), a shift in frequency, but is a dilation of the signal. Thus the return from an object moving at a radial velocity (towards the transmitter/receiver) of v is, leaving aside the magnitude term,

$$\mathrm{ret}(t) = \mathbf{w}\left(\alpha t - \frac{2r}{c+v}\right), \tag{92}$$

where c is the speed of the wave and

$$\alpha = \frac{(1-\frac{v}{c})}{(1+\frac{v}{c})}. \tag{93}$$

This leads, after a rescaling, to an ambiguity function of the form

$$\mathbf{W}_{\mathbf{u},\mathbf{v}}(r,\alpha) = \sqrt{\alpha}\int_{-\infty}^{\infty} \mathbf{u}(t)\overline{\mathbf{v}\big(\alpha(t+r)\big)}\, dt. \tag{94}$$

The relevant representation in the wide band case is the representation

$$\rho_{(a,b)}(\mathbf{v})(t) = \sqrt{a}\,\mathbf{v}(at+b), \quad \mathbf{v} \in \mathsf{L}^2(\mathbf{R}), \tag{95}$$

of the affine group

$$\mathcal{A} = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} :\ a > 0,\ b \in \mathbf{R}\right\} \tag{96}$$

with group multiplication given by matrix multiplication. Note that $\rho_{(a,b)}$ is actually a representation *on the right*, meaning $\rho_{g_1 g_2} = \rho_{g_2}\rho_{g_1}$ for all $g_1, g_2 \in \mathcal{A}$. Analogous to the narrow band case, the wide band ambiguity function can be expressed (after a change of coordinates) as

$$\mathbf{W}_{\mathbf{u},\mathbf{v}}(a,b) = \big\langle \mathbf{u}, \rho_{(a,b)}\mathbf{v}\big\rangle \tag{97}$$

and its properties are studied by investigating the representation $\rho_{(a,b)}$ of the affine group.

This investigation is not undertaken here for reasons of space. The interested reader is, however, referred to (Miller, 1991) for details. It is remarked though that the representation theory of the affine group is complicated by the affine group not being unimodular. That is to say, whereas the left invariant Haar measure defined in (13) of the Heisenberg group is also the right invariant measure of the Heisenberg group, the left and right invariant measures of the affine group differ.

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