Quadrature Mirror Filters, Low Crest Factor Arrays, Functions Achieving Optimal Uncertainty Principle Bounds, and Complete Orthonormal Sequences — a Unified Approach†

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Abstract: Quadrature mirror filters are used throughout image and speech analysis; low crest factor arrays play an important role in many problems in antenna design and communication; uncertainty principle bounds limit resolution in time-frequency analysis and other areas; and complete orthonormal sequences are crucial in essentially all applications of mathematics to science and engineering. We present a single construction which unifies these four concepts.

1. Introduction

We construct a sequence of polynomials $\{P_{n,m}(z)\}$, $n \geq 0$, $0 < m \leq 2^n$, with coefficients ± 1, each of which is a quadrature mirror filter (QMF) and each of which also represents an antenna array with uniformly low crest factor $\sqrt{2}$. Furthermore, by defining $Q_{n,m}(z) = (1-z)P_{n,m}(z^2)$, and then taking the sequence of piecewise constant functions on $(0, 2\pi)$ whose values are the ± 1 coefficients of the $Q_{n,m}$'s, there immediately results a complete orthornormal sequence (CONS) for the space $L^2(0, 2\pi)$ (and, in fact, for $C[0, 2\pi]$). Finally, these "Walsh-like" (but definitely not Walsh!) functions, when combined with wavelet-like dilations and translations, give a CONS for $L^2(\mathbf{R})$ which is optimal with respect to the uncertainty principle.

2. Crest factor

The *crest factor* of an antenna array, sometimes called the *peak factor*, is the ratio of the maximum power output to the *RMS* output of the array. Its importance arises, for instance, in radar and sonar, where the problem often occurs of maximizing average power for a given amplitude range. Another example is

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in communications, where the power spectrum of a periodic signal is given and the problem is to minimize the peak-to-peak amplitude. Minimization of the peak factor is also important in speech synthesis, as this tends to enhance the perceived quality of synthetic speech.

Previous works on the synthesis of low crest factor arrays, including both those in the engineering literature [1, 8, 10, 11, 24, 27, 28, 32] and those presented by mathematicians designing "flat" polynomials [3, 5, 6, 9, 15, 16, 17, 18, 19, 22, 23, 26], have dealt with the construction of individual functions which yield low crest factors. The purpose of this work is to demonstrate a CONS (also called a *basis*) for the space of finite energy functions on a bounded interval, all of which yield low crest factor arrays. Furthermore, we simultaneously obtain an infinite collection of QMFs, all of which have the computationally desirable property that their coefficients are ± 1. Finally, this construction, combined with wavelet-like dilations and translations, immediately gives a sequence which is optimal with respect to the uncertainty principle for infinite orthonormal subsets of the space of finite energy functions on the real line, and thereby affirms a conjecture of H.S. Shapiro.

3. Uncertainty Principles

Classical "uncertainty principles" in Fourier analysis refer to results which assert, for $f \in L^2(\mathbf{R})$, that unless $f = 0$ it cannot happen that *both* f and its Fourier transform $\hat{f}$ are "small" in prescribed ways. For example, f and $\hat{f}$ cannot both be of compact support unless $f = \hat{f} = 0$. More precisely (Hardy), they cannot both be $o(e^{-x^2/2})$ at ∞. There are many variations on this theme, e.g., if f vanishes on a half-line and $\hat{f}$ vanishes on a set of positive measure then $f = 0$, and there are further refinements due to Levinson, Beurling, et al. Although there remain open questions in this area, we may fairly say that uncertainty principles *for a single function* are reasonably well understood.

Here we deal with "uncertainty" for *classes* of functions, i.e., theorems of the form: certain "smallness at ∞" conditions applying to f and $\hat{f}$ *cannot hold uniformly with respect to all the functions f in a class $\mathcal{F}$.* (Typical choices of $\mathcal{F}$ are: an infinite orthonormal subset of $L^2(\mathbf{R})$; an orthonormal basis of $L^2(\mathbf{R})$; a "frame"; or a wavelet basis for this space.) The point being that because smallness is imposed uniformly with regard to an infinite class, the *amount* of smallness required is drastically less than that needed in an uncertainty principle for a single function. Thus, while it happens (for $f(x) = e^{-x^2/2}$) that f and $\hat{f}$ can be nontrivially $O(e^{-x^2/2})$, Shapiro [4] has shown that a well-known compactness result of Kolmogorov yields the impossibility of even

$$|f_n(x)| \le C(1+|x|)^{-p}, \quad |\hat{f}_n(\xi)| \le C(1+|\xi|)^{-p} \tag{3.1}$$

holding, when p is larger than $1/2$, for *all* $\{f_n\}$ in an *infinite orthonormal set* (here C is to be independent of n).

Considering a possible converse, Shapiro made the following basic

Conjecture 3.1. There is an orthonormal basis $\{\phi_n\}$ for $L^2(0, 2\pi)$ such that for some $K < \infty$,

1) $\sup_{n,x} |\phi_n(x)| \le K$;
2) $\left|\int_0^{2\pi} \phi_n(x) e^{-ix\xi}\, dx\right| \le C/\sqrt{1+|\xi|}$

uniformly with respect to n ($\xi \in \mathbf{R}$).

It is straightforward to show that, by properly dilating and shifting these ϕ_n, there results an orthonormal basis for $L^2(\mathbf{R})$ that satisfies (3.1) for $p = 1/2$, so that $\frac{1}{2}$ is indeed the correct breaking point for the value of p in (3.1).

4. New Results

We now present our construction proving the Shapiro conjecture and, simultaneously, producing the QMFs and low crest factor CONS of the title. Namely, we have the

Theorem 4.1. There exists an orthonormal basis $\{\phi_n\}$ for $L^2(0, 2\pi)$, and in fact for $C[0, 2\pi]$, such that

1) For every n and x, $\phi_n(x)$ takes on only the values ± 1,
2)
$$\left|\int_0^{2\pi} \phi_n(x) e^{-ix\xi}\, dx\right| \le \frac{18}{\sqrt{\pi}\sqrt{1+|\xi|}}$$
uniformly in n, and
3) The polynomials $Q_{n,m}(z)$ defined below, whose coefficients make up these $\{\phi_n\}$'s, satisfy
$$\frac{\mathrm{Max}_{|z|=1} |Q_{n,m}(z)|}{\|Q_{n,m}(z)\|_{L^2}} \le 2$$
(note that this ratio is the crest factor).

We begin with the following lemma.

Lemma 4.2. Let $\{T_k(z)\}_{k=1}^{\infty}$ be a sequence of "semiflat unimodular polynomials" of degree $k-1$ with coefficients $c_j(k)$, $0 \le j \le k-1$. That is,

$$T_k(z) = \sum_{j=0}^{k-1} c_j(k) z^j, \qquad |c_j(k)| = 1 \text{ for all } j \text{ and } k, \tag{4.1}$$

and there is an absolute positive constant M such that

$$|T_k(z)| \le M\sqrt{k} \qquad \text{for all } k \text{ and } |z| = 1. \tag{4.2}$$

(Note: If there was also an absolute positive constant $M^{\star}$, necessarily less than 1, such that $|T_k(z)| \ge M^{\star}\sqrt{k}$ for all k and $|z| = 1$, then the T_k's would be called "flat." Although the existence of such polynomials has been demonstrated probabilistically [15, 16], a well-known open problem in mathematical analysis is to actually construct them.)

If we then define, for each k, the piecewise constant function $g_k(x)$ by

$$g_k(x) = \begin{cases} c_j(k), & 2\pi\frac{j}{k} \le x < 2\pi\frac{j+1}{k},\ 0 \le j \le k-1 \\ c_{k-1}(k), & x = 2\pi \\ 0, & x \notin [0, 2\pi] \end{cases}$$

we have $|g_k(x)| = 1$ for all $k \ge 1$ and $x \in [0, 2\pi]$, and

$$\left| \int_0^{2\pi} g_k(x) e^{-ix\xi}\, dx \right| \le \frac{9M}{\sqrt{1+|\xi|}} \qquad \text{for all } k \ge 1 \text{ and } \xi \in \mathbf{R}.$$

Proof. Clearly $|g_k(x)| \equiv 1$ on $[0, 2\pi]$. Let

$$\begin{aligned} A_k \equiv \int_0^{2\pi} g_k(x) e^{-ix\xi}\, dx &= \sum_{j=0}^{k-1} c_j(k) \int_{2\pi\frac{j}{k}}^{2\pi\frac{j+1}{k}} e^{-ix\xi}\, dx \\ &= \sum_{j=0}^{k-1} c_j(k) \frac{i}{\xi} e^{-ix\xi} \Bigg]_{2\pi\frac{j}{k}}^{2\pi\frac{j+1}{k}} = \frac{i}{\xi} \sum_{j=0}^{k-1} c_j(k) [e^{-2\pi i\xi\frac{j+1}{k}} - e^{-2\pi i\xi\frac{j}{k}}] \\ &= \frac{i}{\xi}(e^{-2\pi i\frac{\xi}{k}} - 1) \sum_{j=0}^{k-1} c_j(k) (e^{-2\pi i\frac{\xi}{k}})^j = \frac{i}{\xi}(e^{-2\pi i\frac{\xi}{k}} - 1) T_k(z), \quad z = e^{-2\pi i\frac{\xi}{k}}. \end{aligned}$$

Now,

$$\begin{aligned}\left|\frac{i}{\xi}\left(e^{-2\pi i\frac{\xi}{k}}-1\right)\right|^2 &= \frac{1}{\xi^2}\left(1-e^{-2\pi i\frac{\xi}{k}}\right)\left(1-e^{2\pi i\frac{\xi}{k}}\right)\\ &= \frac{2}{\xi^2}\left(1-\cos 2\pi\frac{\xi}{k}\right)\\ &< \frac{2}{\xi^2}\frac{4\pi^2\xi^2}{2k^2} = \frac{4\pi^2}{k^2},\end{aligned}$$

since $1-\cos t < t^2/2$.

Thus,

$$\left|\frac{i}{\xi}\left(e^{-2\pi i\frac{\xi}{k}}-1\right)\right| < \frac{2\pi}{k}$$

and

$$|T_k(z)| \le M\sqrt{k} \qquad \forall \xi, k,$$

so that

$$|A_k| < \frac{2M\pi}{\sqrt{k}} \qquad \forall \xi, k. \tag{4.3}$$

Also, trivially,

$$|A_k| < \frac{2M\sqrt{k}}{|\xi|} \qquad \forall \xi, k. \tag{4.4}$$

For $k > \pi|\xi|$, (4.3) yields

$$|A_k| < \frac{2M\sqrt{\pi}}{\sqrt{|\xi|}},$$

while for $k \le \pi|\xi|$, we get from (4.4) that

$$|A_k| < \frac{2M\sqrt{\pi}}{\sqrt{|\xi|}}.$$

Now, for $|\xi| \ge 1$,

$$\frac{1}{\sqrt{|\xi|}} \le \frac{\sqrt{2}}{\sqrt{1+|\xi|}},$$

so that

$$|A_k| < \frac{2^{3/2}M\sqrt{\pi}}{\sqrt{1+|\xi|}} < \frac{6M}{\sqrt{1+|\xi|}} \qquad \forall k \ge 1 \text{ and } |\xi| \ge 1.$$

It remains to consider the case $|\xi| < 1$. But here we just apply the trivial bound $|A_k| \le 2\pi$ and the obvious fact that $M \ge 1$ to get

$$|A_k| \le 2\pi < \frac{2\sqrt{2}\pi}{\sqrt{1+|\xi|}} \le \frac{2\sqrt{2}\pi M}{\sqrt{1+|\xi|}} < \frac{9M}{\sqrt{1+|\xi|}},$$

and the lemma is proved. ∎

The remainder of the proof is based upon the well-known and remarkable Shapiro Polynomials, introduced by H.S. Shapiro in his 1951 master's thesis [29], and published in 1959 by Rudin [25]. These are defined inductively as follows:

$$P_0(z) = Q_0(z) = 1$$

and for $k \geq 0$,

$$\begin{aligned} P_{k+1}(z) &= P_k(z) + z^{2^k} Q_k(z), \\ Q_{k+1}(z) &= P_k(z) - z^{2^k} Q_k(z), \end{aligned} \tag{4.5}$$

It is immediate that for all $|z| = 1$ and all $k \geq 0$,

$$|P_{k+1}(z)|^2 + |Q_{k+1}(z)|^2 \equiv 2(|P_k(z)|^2 + |Q_k(z)|^2) \equiv 2^{k+2}. \tag{4.6}$$

It is also clear that P_k and Q_k are each polynomials of degree $2^k - 1$, with coefficients ± 1. Thus, $\{P_k\}$ and $\{Q_k\}$ are each sequences of semiflat unimodular polynomials with $M = \sqrt{2}$. Our idea is to choose the $c_j(k)$'s as coefficients of variations of the Shapiro Polynomials, obtaining the required orthogonality and completeness while preserving the flatness and the unimodularity.

In what follows, and also in the proof of Lemma 4.3, by "orthogonality" we mean that the piecewise constant functions $g_k(x)$ of the lemma are orthogonal. Thus this notion of orthogonality is quite different from the usual orthogonality of polynomials on the unit circle. As a first step, for each $n \geq 1$, we construct a set of 2^n piecewise orthogonal Shapiro-type polynomials $P_{n,m}(z)$, $m = 1, 2, \ldots, 2^n$, with each $P_{n,m}$ of degree $2^n - 1$. To begin this double induction, let $P_{1,1}(z) = P_1(z)$ and $P_{1,2}(z) = Q_1(z)$. Since the coefficient set of $P_1(z)$ is $\{1, 1\}$, while that of $Q_1(z)$ is $\{1, -1\}$, $P_{1,1}$ and $P_{1,2}$ are clearly orthogonal. Next, let $P_{2,1}(z) = P_2(z) = P_{1,1} + z^2 P_{1,2}$, $P_{2,2}(z) = Q_2(z) = P_{1,1} - z^2 P_{1,2}$, $P_{2,3}(z) = P_{1,2} + z^2 P_{1,1}$, and $P_{2,4} = P_{1,2} - z^2 P_{1,1}$. $P_{2,1}$ is orthogonal to $P_{2,2}$ for the same reason that $P_{2,3}$ is orthogonal to $P_{2,4}$: namely, the sum of the squares of the coefficients of $P_{1,1}$ equals the sum of the squares of the coefficients of $P_{1,2}$ $(= 2)$. The other required orthogonality relations, $P_{2,1} \perp P_{2,3}$, $P_{2,1} \perp P_{2,4}$, $P_{2,2} \perp P_{2,3}$, and $P_{2,2} \perp P_{2,4}$ follow immediately from the fact that $P_{1,1} \perp P_{1,2}$.

The pattern of the construction is now clear. Given $P_{n,m}(z)$, $m = 1, 2, \ldots, 2^n$, for $j = 0, 1, 2 \ldots, 2^{n-1} - 1$ and $m = 4j + 1$ define $P_{n+1,m}(z)$, $m = 1, 2, \ldots, 2^{n+1}$ by:

$$\begin{aligned} P_{n+1,m} &= P_{n,2j+1} + z^{2^n} P_{n,2j+2} \\ P_{n+1,m+1} &= P_{n,2j+1} - z^{2^n} P_{n,2j+2} \\ P_{n+1,m+2} &= P_{n,2j+2} + z^{2^n} P_{n,2j+1} \\ P_{n+1,m+3} &= P_{n,2j+2} - z^{2^n} P_{n,2j+1} \end{aligned} \tag{4.7}$$

For example, for $n = 2$, we get

$$
\begin{aligned}
P_{3,1} &= P_{2,1} + z^4 P_{2,2} \\
P_{3,2} &= P_{2,1} - z^4 P_{2,2} \\
P_{3,3} &= P_{2,2} + z^4 P_{2,1} \\
P_{3,4} &= P_{2,2} - z^4 P_{2,1} \\
P_{3,5} &= P_{2,3} + z^4 P_{2,4} \\
P_{3,6} &= P_{2,3} - z^4 P_{2,4} \\
P_{3,7} &= P_{2,4} + z^4 P_{2,3} \\
P_{3,8} &= P_{2,4} - z^4 P_{2,3}
\end{aligned}
$$

While the sequence for $n = 3$ is

$$
\begin{aligned}
P_{4,1} &= P_{3,1} + z^8 P_{3,2} \\
P_{4,2} &= P_{3,1} - z^8 P_{3,2} \\
P_{4,3} &= P_{3,2} + z^8 P_{3,1} \\
P_{4,4} &= P_{3,2} - z^8 P_{3,1} \\
P_{4,5} &= P_{3,3} + z^8 P_{3,4} \\
P_{4,6} &= P_{3,3} - z^8 P_{3,4} \\
P_{4,7} &= P_{3,4} + z^8 P_{3,3} \\
P_{4,8} &= P_{3,4} - z^8 P_{3,3} \\
P_{4,9} &= P_{3,5} + z^8 P_{3,6} \\
P_{4,10} &= P_{3,5} - z^8 P_{3,6} \\
P_{4,11} &= P_{3,6} + z^8 P_{3,5} \\
P_{4,12} &= P_{3,6} - z^8 P_{3,5} \\
P_{4,13} &= P_{3,7} + z^8 P_{3,8} \\
P_{4,14} &= P_{3,7} - z^8 P_{3,8} \\
P_{4,15} &= P_{3,8} + z^8 P_{3,7} \\
P_{4,16} &= P_{3,8} - z^8 P_{3,7}
\end{aligned}
$$

For each $n \geq 1$, the pairwise orthogonality of $P_{n,m}$, $1 \leq m \leq 2^n$, obviously follows.

Moreover, the flatness of the Shapiro Polynomial is preserved in this generalization. In fact, we have the next lemma.

Lemma 4.3. For each $n \geq 1$, each $j = 0, 1, 2, \ldots, 2^{n-1} - 1$, and each $m = 4j + 1$,

$$|P_{n+1,m}|^2 + |P_{n+1,m+1}|^2 \equiv |P_{n+1,m+2}|^2 + |P_{n+1,m+3}|^2 \equiv 2^{n+2}, \qquad (4.8)$$

so that all such $P_{n,m}(z)$ are semiflat unimodular polynomials with $M = \sqrt{2}$.

Proof. For $n = 1$ this follows immediately from the definition of these polynomials and the corresponding properties of the Shapiro Polynomials $P_2(z)$ and $Q_2(z)$. Equations (4.7) trivially yield the necessary induction step, proving the lemma.

As the alert reader has undoubtedly observed, we have pairwise orthogonality of the $P_{n,m}$'s for each n, but *not* across n's, so that one last trick is required. However, this difficulty is more apparent than real, as it can be overcome by the following simple device. Namely, define for each $P_{n,m}(z)$ above the polynomial $Q_{n,m}(z) = (1 - z)P_{n,m}(z^2)$. By this interspersing of the negatives of the coefficients of $P_{n,m}$ with the coefficients themselves, the required orthogonality "across n's" is clearly achieved, while for the same n, the previous orthogonality "across m's" is obviously preserved. Moreover, the flatness remains, as all such $Q_{n,m}(z)$ are obviously semiflat unimodular polynomials with $M = 2$.

Completeness is the only remaining consideration. Defining $Q_{0,0}(z) = 1$ and $Q_{0,1}(z) = 1 - z$, for each N we have a total of 2^N pairwise orthogonal (in the above sense) polynomials $Q_{n,m}(z)$. That is, returning to the notation of the beginning of the paper, we have 2^N mutually orthogonal functions $g_k(x)$, $0 \leq k \leq 2^N - 1$, each of which is piecewise constant on all subintervals $[2\pi(j-1)/2^N, 2\pi j/2^N)$, $1 \leq j \leq 2^N$, and takes on only the values ± 1 on these subintervals. It follows from elementary linear algebra that these $g_k(x)$, $0 \leq k \leq 2^N - 1$, span the set of 2^N functions $f_k(x)$ given by

$$f_k(x) = \begin{cases} 1, & 2\pi \frac{k}{2^N} \leq x < 2\pi \frac{k+1}{2^N}, \\ 1, & k = 2^N - 1, x = 2\pi, \\ 0, & \text{otherwise}. \end{cases}$$

As the closed span of the set of all such f_k is $L^2(0, 2\pi)$ in the L^2 norm, or $C[0, 2\pi]$ in the uniform (L^∞) norm, we see finally that the sequence

$$\left\{ \frac{1}{\sqrt{2\pi}} g_k(x) \right\}_{k=0}^{2^N - 1}, \qquad N \geq 0$$

forms a complete orthonormal set for $L^2(0, 2\pi)$ and for $C[0, 2\pi]$, and the theorem is proved. ∎

5. Quadrature mirror filters

A discrete filter $\{h_n\}_{n\in\mathbb{Z}}$ whose transfer function $H(z)$ satisfies

$$H(z)H(1/z) + H(-z)H(-1/z) = \text{constant for } |z| = 1 \tag{5.1}$$

is a *quadrature mirror filter* (QMF). First introduced by Croisier, Esteban, and Galand [13, 14], QMFs were initially used in the subband coding of speech. More recently, their properties of aliasing cancellation and low frequency distortion have led to the extensive application of QMFs in subband image coding. This was first proposed by Vetterli [34], and has been extended by Vaidyanathan [33], Mallat [20, 21], and many others. [30] and [31] are excellent references in this regard.

It has apparently escaped notice until now that the Shapiro Polynomials $P_k(z)$ and $Q_k(z)$ defined in (4.5) are QMFs. This follows immediately from (4.6) and from the following identities, easily proven by induction:

$$\begin{aligned} P_n(-z) &= (-1)^n z^{2^n-1} Q_n(1/z) \\ Q_n(-z) &= (-1)^{n+1} z^{2^n-1} P_n(1/z) \end{aligned} \tag{5.2}$$

The identities (5.2) and similar results were first proved in H.S. Shapiro's Sc.M. thesis [29] (which was unpublished and therefore little known). They were subsequently rediscovered by several authors. Since our generalizations of the Shapiro Polynomials, the $P_{n,m}(z)$ given by (4.7), satisfy (4.8) as well as similar identities to (5.2), it follows that

Theorem 5.1. All polynomials $P_{n,m}(z)$ given by (4.7) are QMFs.

Note that these newly-constructed QMFs, the $P_{n,m}(z)$, have coefficients which are all ± 1. This should aid significantly in the computational aspects of their applications to speech, image processing, and other areas.

6. Conclusion

Each polynomial in the above sequence $\{Q_{n,m}(z)\}$, $n \geq 0$, $1 \leq m \leq 2^n$, has coefficients ± 1 and represents an array with crest factor at most 2 (independent of n). The piecewise constant functions $g_k(x)$ given above, whose values are simply the coefficients of these Q's, form a CONS for the space of finite energy functions on $(0, 2\pi)$, and so satisfy all of the basic properties of the Walsh functions [2, 7, 12, 35]. Furthermore, these g's are optimal with respect to the uncertainty principle for infinite orthonormal subsets of the space of finite energy functions on the real line.

As polynomials of degree $2^N - 1$ employing the Walsh functions as coefficients have the worst possible crest factor, $\sqrt{2^N}$, and as these functions do not satisfy the uncertainty principle properties described above, it is natural to ask whether our g_k's may be applied in the various areas of coding and signal processing where Walsh functions have been successfully employed. Research on this aspect of our work is currently underway.

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